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Well-posedness study for a time-domain spherical cloaking model

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ABSTRACT

In this paper, we study a time-domain spherical cloaking model recently introduced by Zhao and Hao (2009). This model is quite complicated and is composed of four coupled differential equations. Here we first prove the existence and uniqueness of a solution for this model. Then we obtain a stability result, which analysis is quite involved due to the coupling between the four variables. To our best knowledge, this is the first well-posedness study carried out for the time-domain cloaking model with metamaterials.

1. Introduction

The idea of invisibility cloaking using metamaterials started in 2006 when Pendry et al. [1] and Leonhardt [2] laid out the blueprints for making objects invisible to electromagnetic waves. In late 2006, a 2-D reduced cloak was successfully fabricated and demonstrated to work at 8.5 GHz and relied upon local resonances of split ring resonators [3]. This is the first practical realization of such a cloak, and the result matches well with the computer simulation [4] performed using the commercial package COMSOL. The cloaking technique of [1,2] is to establish a correspondence between physical material parameters (the material's permittivity and permeability) and coordinate transformations. The conceptual device constructed with these material parameters is able to guide waves to propagate around the cloaked region (usually the central region of the cloaking structure), and render the objects placed inside invisible to external electromagnetic radiations. It turns out that essentially the same idea was discussed earlier in 2003 by Greenleaf, Lassas, and Uhlmann [5,6] for electrical impedance tomography. Now this transformation technique is widely used in various cloak designs, and has earned a variety of names such as Transformation Electromagnetics, Transformation Optics, Transformation Acoustics, and Transformation Elastodynamics (see recent review papers [7–10] and the book [11]).

In addition to the transformation optics (and acoustics) technique, there are many different avenues towards electromagnetic and acoustic cloaking. Another kind of cloaking [12,13] requires a negative refractive index shell, which allows for the cloaking of a discrete set of dipoles when they are located within a given distance outside the shell. A third kind of cloaking uses complementary media to cloak objects at a distance outside the cloaking shell [14]. A fourth kind of cloaking is obtained via active scattering cancellation devices not completely surrounding the cloaked region (exterior cloaking) [15,16]. A very recent cloaking technique is to use zero index metamaterials loaded with normal dielectric defects [17,18].

Since 2006, study of using metamaterials to construct invisibility cloaks has been a very hot research topic. A search on "metamaterials and cloaking" over scholar.google.com (conducted on April 28, 2014) shows 1880 publications since

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2013. Most of them focus on engineering and physics. Compared to the huge amount of papers published in engineering and physics, there are not much mathematical analyses done for metamaterials and cloaking, even though numerical simulation in metamaterials [19,20] plays a very important role in cloaking structure design and validation of the theoretical predictions.

In recent years, mathematicians have started investigating this fascinating subject, but most works are still limited to frequency-domain or the quasi-static regime by mainly solving the Helmholtz equation [21–27], and the time-harmonic Maxwell's equations [28,29]. The advancement of broadband cloaks [30,31] makes time-domain cloaking simulation more appealing and necessary. Generally speaking, electromagnetic wave cloaking simulation boils down to solving metamaterial Maxwell's equations in either frequency-domain or time-domain. In 2012, we developed the first time-domain finite element method to simulate a cylindrical cloak [32], and completed the well-posedness study of this model in [33]. In this work, we carry out a rigorous analysis of the well-posedness for a time-domain spherical cloaking model recently developed and simulated by using the FDTD method [34]. Though there exist a few publications on well-posedness for metamaterial Maxwell's equations in frequency-domain (e.g. [35–37]) and time-domain [38], to the best of the author's knowledge, we are unaware of other works on the well-posedness study of time-domain cloaking models. The major challenge for the analysis is that this model is quite complicated, and is formed by four mixed order differential equations with four vector unknowns.

The rest of the paper is organized as follows. In Section 2, we provide a detailed derivation of the time-domain spherical cloak modeling equations, since the original paper does not even present the complete set of governing equations. Then in Section 3, we first prove the existence and uniqueness of our model problem, then we prove the stability of the model. We conclude the paper in Section 4.

2. The modeling equations

The permittivity and permeability of the ideal spherical cloak are given by [1]:

$$\epsilon_r = \mu_r = \frac{R_2}{R_2 - R_1}\left(\frac{r - R_1}{r}\right)^2, \quad R_1 \le r \le R_2, \tag{1}$$

$$\epsilon_\theta = \mu_\theta = \frac{R_2}{R_2 - R_1}, \qquad \epsilon_\phi = \mu_\phi = \frac{R_2}{R_2 - R_1}, \tag{2}$$

where R_1 and R_2 are the inner and outer radii of the cloak, and r denotes the radial distance from the center of the cloak.

Due to the inconvenience of cloaking simulation in spherical coordinate (e.g., COMSOL, the popular simulation software in this area, is only for Cartesian coordinate), the permittivity and permeability parameters given above have to be changed to Cartesian coordinate via the following transformation [34]:

$$\begin{aligned}\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix} &= \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\epsilon_r & 0 & 0\\ 0 & \epsilon_\theta & 0\\ 0 & 0 & \epsilon_\phi\end{bmatrix}\\ &\quad\times\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}.\end{aligned} \tag{3}$$

Substituting (3) into the constitutive equation

$$\epsilon_0\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix} = \begin{bmatrix}D_x\\ D_y\\ D_z\end{bmatrix},$$

we have (details see [34]):

$$\begin{aligned}\epsilon_0 E_x &= \left(\frac{1}{\epsilon_r}\sin^2\theta\cos^2\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\cos^2\phi + \frac{1}{\epsilon_\phi}\sin^2\phi\right)D_x\\ &\quad + \left(\frac{1}{\epsilon_r}\sin^2\theta\sin\phi\cos\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin\phi\cos\phi - \frac{1}{\epsilon_\phi}\sin\phi\cos\phi\right)D_y\\ &\quad + \left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\cos\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\cos\phi\right)D_z,\end{aligned} \tag{4}$$

$$\begin{aligned}\epsilon_0 E_y &= \left(\frac{1}{\epsilon_r}\sin^2\theta\sin\phi\cos\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin\phi\cos\phi - \frac{1}{\epsilon_\phi}\sin\phi\cos\phi\right)D_x\\ &\quad + \left(\frac{1}{\epsilon_r}\sin^2\theta\sin^2\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin^2\phi + \frac{1}{\epsilon_\phi}\cos^2\phi\right)D_y\\ &\quad + \left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\sin\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\sin\phi\right)D_z,\end{aligned} \tag{5}$$