Computers and Mathematics with Applications 68 (2014) 1915–1932

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analysis of the thin film flow in a rough domain filled with micropolar fluid

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ARTICLE INFO

Article history:
Received 24 March 2014
Received in revised form 2 October 2014
Accepted 3 October 2014
Available online 18 October 2014

Keywords:
Thin-film flow
Micropolar fluid
Rough boundary
Different scales
Asymptotic expansion
Two-scale convergence

ABSTRACT

Inspired by the lubrication framework, in this paper a micropolar fluid flow through a rough thin domain is studied. The domain's thickness is considered as the small parameter ε, while the roughness is defined by a periodical function with period of order ε^2. Starting from three-dimensional micropolar equations and using asymptotic analysis with respect to ε, we formally derive the macroscopic model clearly detecting the effects of the specific rugosity profile and fluid microstructure. We provide the rigorous justification of our formally obtained asymptotic model by deriving the effective system by means of the two-scale convergence.

1. Introduction

The classical lubrication problem is mainly concerned with the situation in which two solid surfaces being in relative motion are separated by a thin layer of fluid acting as a lubricant. Such situation appears naturally in applications consisting of moving machine parts, namely the journal bearings. Fluid film bearings are machine elements whose function is to promote smooth relative motion between two surfaces and are crucial factors in limiting the dissipation of energy. The ultimate goal is that fluid film bearing is well designed so that the wear is not an issue (two surfaces are completely separated by the lubricant). For that reason, it is essential to understand the behavior of the fluid film in such machine elements. The first result goes back to Reynolds and his celebrated work [1] published in 1886. He studied the thin film flow in a rather heuristic manner and did not provide any relation between his model and the Navier–Stokes equations. The formal relationship between Navier–Stokes equations and Reynolds equation in a thin domain was established more than 60 years later in [2,3], while the rigorous mathematical justification of the Reynolds equation for a Newtonian flow between two plain surfaces can be found in [4].

If the gap between the moving surfaces becomes very small, the experimental results from the tribology literature (see e.g. [5–7]) suggest that the fluid's internal structure should be taken into account as well. A possible way to acknowledge such experimental findings is to employ the micropolar fluid model. Being originally proposed by Eringen [8] in the 60s, the theory of micropolar fluids has gained much attention since it successfully describes the effects of local structure and micro-motions of the fluid elements that cannot be captured by the classical Navier–Stokes model. Physically, micropolar fluids represent fluids consisting of rigid, spherical particles suspended in a viscous medium, where the deformation of fluid

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http://dx.doi.org/10.1016/j.camwa.2014.10.003
0898-1221/

particles is ignored. They are, in fact, non-Newtonian fluids with nonsymmetric stress tensor. In view of that, the related mathematical model introduces a new vector field, the angular velocity field of rotation of particles (microrotation) and one new (vector) equation coming from the conservation of the angular momentum. As a result, a complex coupled system of PDEs is obtained, representing a significant generalization of the Navier–Stokes equations. We refer the reader to the monograph [9] (and the references therein) providing a detailed derivation of the micropolar equations from the general constitutive laws together with an extensive review of the mathematical theory and the applications of this particular model.

Engineering practice also indicates that it is of great interest to combine the lubrication phenomena with the analysis of the roughness effects. Usually it means that the lower surface is assumed to be perfectly smooth, but the upper is rough and described by a given function. Expressing the boundary roughness using a periodic function, thin-film flow of Newtonian fluid has been extensively studied for different rugosity profiles. The classical assumption is that the size of the roughness is of the same order as the film thickness, i.e.

$$h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon}\right), \quad 0 < \varepsilon \ll 1. \tag{1}$$

In such setting, the effective model turns out to be the classical Reynolds equation (see e.g. [10,11]) and one needs to compute the correctors in order to detect the roughness-induced effects. Same result is obtained for $h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^\beta}\right)$ with $\beta < 1$ (see [12]). In view of that, Bresch and co-authors [13] in 2010 considered a new framework, namely

$$h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^2}\right). \tag{2}$$

As a result, they derived the asymptotic model in which an extra term (appearing due to the boundary roughness) modifies the standard Reynolds equation at the main order. Whole asymptotic expansion (at any order) of the solution has been rigorously derived in [14] providing the optimality with respect to the truncation error. It is important to emphasize that, roughness pattern described by $h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^\beta}\right)$ with $\beta > 1$ is physically relevant and realistic (see e.g. [15]), and, therefore, has been studied for different situations in recent years. Focusing on the wall laws, the effects of the above setting on the asymptotic behavior of the Navier–Stokes system have been investigated in [16]. Using the asymptotic approximation from [13] derived for the hydrodynamic part of the system, the roughness effects on the heat conduction in a thin film flow have been studied in [17]. A semilinear parabolic problem in a thin rough domain assuming different order to the period of oscillations on the top and the bottom of the boundary has been addressed in [18].

Our goal is to extend the analysis presented in [13] to a case of lubrication with incompressible micropolar fluid. There are not many papers in the existing literature dealing with the mathematical modeling of micropolar fluid film lubrication. Interesting result can be found in [19] where the authors consider a specific slider-type bearing. After writing the governing problem in non-dimensional form, they formally obtain a generalized version of the Reynolds equation in a critical case when one of the non-Newtonian characteristic parameters has specific (small) order of magnitude. Rigorous derivation of such result was brought 14 years later in [20] for two-dimensional setting (see also [21] for micropolar flow in a curved channel). The 3D lubrication problem was recently addressed in [22] and new, second-order Brinkman-type asymptotic model has been proposed. In the above papers, the roughness effects were not taken into account, i.e. the height of the channel is assumed to be of the form $h_\varepsilon(x) = \varepsilon\, h(x)$. To our knowledge, the first (and only) rigorous result on the micropolar fluid film lubrication in a thin domain with rough boundary can be found in the recent paper by Boukrouche and Paoli [23]. They consider a micropolar flow in a two-dimensional domain assuming that the height of the channel is given by (1). Employing two-scale convergence technique, they derive the limit problem describing the macroscopic flow. In the present paper, we are going to study a micropolar fluid flow in a three-dimensional domain given by

$$\Omega_\varepsilon = \left\{(x, z) \in \mathbf{R}^2 \times \mathbf{R} : x \in \omega,\ 0 < z < h_\varepsilon(x)\right\}, \tag{3}$$

where the height h_ε is defined by (2). From the point of view of asymptotic analysis, we find this framework more challenging than the classical one (given by (1)) due to the technical difficulties caused by the specific height profile.

The main problem related to a fluid flow through a domain with roughness is to deduce in which way the irregular boundaries affect the flow. This is especially important with regard to numerical computations: indeed, roughness is in general too small to be captured by the discretization grid of the simulations. To overcome this difficulty, one can employ the homogenization theory. In view of that, the idea is to replace the irregular domain by a smooth one, and then describe the averaged effect of the roughness in the limit (homogenized) model. For that reason, homogenized models have been of practical interest in numerical codes. In our particular case, starting with original problem (6)–(11) posed in thin rough domain Ω_ε, we apply the suitable change of variables, namely $Z = z/h^\varepsilon(x)$, to transform Ω_ε into Ω which is smooth. Then by means of a two-scale convergence technique, we obtain the simplified limit problem posed in Ω, in which the effects of roughness can be clearly observed (see Section 2.3).

The paper is organized as follows. After formulating the problem in Section 2, in Section 3 we perform a formal asymptotic analysis with respect to the small parameter ε. Introducing a suitable change of variables which takes into account the rough oscillations, we rewrite the governing problem in the ε-independent domain and employ two-scale expansion technique. Since the problem is coupled, we construct the asymptotic expansion of the solution by simultaneously treating boundary-value problems for velocity and for microrotation. As a result, we obtain an effective system describing the macroscopic flow and observing clearly the effects of the rugosity profile and fluids microstructure.