Computers and Mathematics with Applications 68 (2014) 2028–2035

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Nonexistence of global solutions for a class of two-time nonlinear evolution equations

Mohamed Jleli [a], Mokhtar Kirane [b,c], Bessem Samet [a,*]

[a] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[b] Laboratoire de Mathématiques, Image et Applications, Pôle Sciences et Technologies, Université de La Rochelle, Avenue M. Crépeau, 17042 La Rochelle, France

[c] NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

ARTICLE INFO

Article history:
Received 7 May 2014
Received in revised form 25 August 2014
Accepted 28 September 2014
Available online 23 October 2014

Keywords:
Ultraparabolic equations
Global solution
Fractional Laplacian
Porous media

ABSTRACT

Nonexistence results of global solutions for a class of ultraparabolic equations and systems involving the fractional Laplacian operator are derived. The proofs are based on the choice of a suitable test function in the weak formulation of the considered problems. The considered equations and systems have their applications in diffusion theory in porous media.

1. Introduction

In this paper, we will study first nonexistence results of global solutions for the two-time nonlinear equation

$$u_{t_1} + u_{t_2} + (-\Delta)^{\alpha/2}(|u|^m) = t_1^s t_2^\ell |x|^r |u|^p, \quad (t_1, t_2, x) \in Q,\ x \neq 0, \tag{1.1}$$

where $Q = (0, \infty) \times (0, \infty) \times \mathbb{R}^N$, $0 < \alpha \leq 2$, $m \geq 1$, $p > 1$, $s \geq 0$, $\ell \geq 0$, $r \geq 0$, and subject to the initial conditions

$$u(t_1, 0, x) = \varphi_1(t_1, x), \qquad u(0, t_2, x) = \varphi_2(t_2, x). \tag{1.2}$$

Here $(-\Delta)^{\alpha/2}$ is the $(\alpha/2)$-fractional power of the Laplacian $-\Delta_x$ in the x variable; it is defined by

$$(-\Delta)^{\alpha/2} v(t_1, t_2, x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(v)(\xi))(t_1, t_2, x), \quad v \in \mathcal{S}',$$

where $\mathcal{S}'$ is the space of Schwartz, $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

Then we will extend our results to systems of the form

$$\begin{cases} u_{t_1} + u_{t_2} + (-\Delta)^{\alpha/2}(|u|^m) = t_1^s t_2^\ell |x|^r |v|^p, \\ v_{t_1} + v_{t_2} + (-\Delta)^{\alpha/2}(|v|^n) = t_1^\sigma t_2^\rho |x|^d |u|^q, \end{cases} \tag{1.3}$$

* Corresponding author.
E-mail addresses: jleli@ksu.edu.sa (M. Jleli), mkirane@univ-lr.fr (M. Kirane), bsamet@ksu.edu.sa (B. Samet).

http://dx.doi.org/10.1016/j.camwa.2014.09.017
0898-1221/

for $(t_1, t_2, x) \in Q, x \neq 0$, where $0 < \alpha \leq 2, m \geq 1, n \geq 1, p > 1, q > 1, s \geq 0, \ell \geq 0, r \geq 0, \sigma \geq 0, \rho \geq 0, d \geq 0$, and subject to the initial conditions

$$\begin{cases} u(t_1, 0, x) = \varphi_1(t_1, x), \qquad u(0, t_2, x) = \varphi_2(t_2, x), \\ v(t_1, 0, x) = \psi_1(t_1, x), \qquad v(0, t_2, x) = \psi_2(t_2, x). \end{cases} \tag{1.4}$$

The considered equation and system have their applications in diffusion theory in porous media.

Ultraparabolic equations known also as multitime parabolic equations are encountered for instance in the theory of Brownian motion (diffusion process with inertia) [1], transport theory (Fokker–Planck type equations) [2], biology (age-structured population dynamics) [3], waves and Maxwell's equations [4], and other practical applications of mathematical physics and engineering sciences. For more details about Ultraparabolic equations, we refer to [5–11] and references therein.

In [12], Fujita considered the Cauchy problem

$$u_t - \Delta(u) = |u|^p, \quad t > 0,\ x \in \mathbb{R}^N.$$

He proved that if $1 < p < 1 + 2/N$, then no global positive solutions for any nonnegative initial data u_0 exist. However, if $p > 1+2/N$, then global small data solutions exist while global solutions for large data do not exist. The case $p = 1+2/N$ has been studied by Hayakawa [13] for $N = 1, 2$ and then by Kobayashi et al. [14] for any $N \geq 1$. The exponent $p_{\text{crit}} = 1 + 2/N$ is called the critical exponent.

Samarskii et al. [15] considered the more general equation

$$u_t - \Delta(u^m) = |u|^p, \quad t > 0,\ x \in \mathbb{R}^N.$$

They showed that the critical exponent is $p_{\text{crit}} = m + 2/N$.

For other related works, we refer to [16–22] and references therein.

Kerbal and Kirane [5] established nonexistence results to the ultraparabolic equation

$$u_{t_1} + u_{t_2} - \Delta(|u|^m) = |u|^p, \quad t_1 > 0,\ t_2 > 0,\ x \in \mathbb{R}^{\mathbb{N}}.$$

They also considered the case of a system having the following form

$$\begin{cases} u_{t_1} + u_{t_2} - \Delta(|u|^m) = |v|^p, \\ v_{t_1} + v_{t_2} - \Delta(|v|^n) = |u|^q. \end{cases}$$

The aim of this paper is to obtain sufficient conditions for the nonexistence of nontrivial global solutions to Eq. (1.1) and to system (1.3). The proofs of our results are based on the weak formulation of the solution with a suitable choice of the test function (see [23]).

Before presenting our results, let us dwell briefly on our choice of the considered nonlinearities. A general physical nonlinearity should be

$$f(x, t, u, \nabla u, \lambda_1, \lambda_2, \ldots),$$

where $\lambda_1, \lambda_2, \ldots$ are possible parameters. The form of the nonlinearity is usually obtained via empirical laws like the mass action law of Guldberg and Waage or Arrhenius law [24]. A prototype nonlinearity can be

$$f(x, t, u) = \varphi(x, t)\psi(u),$$

where φ and ψ are polynomial functions. A simple choice is

$$\begin{cases} \varphi(x, t) = t^a |x|^b, \\ \psi(u) = u^p. \end{cases}$$

2. Results

2.1. The case of a single equation

Solutions to (1.1) subject to conditions (1.2) are meant in the following weak sense.

For $T > 0$, let $Q_T = (0, T) \times (0, T) \times \mathbb{R}^N$ and $S_T = (0, T) \times \mathbb{R}^N$.

Definition 2.1. A function $u \in L^m_{\text{loc}}(Q_T)$ is a local weak solution to (1.1)–(1.2) defined in Q_T, if $h^{1/p}u \in L^p_{\text{loc}}(Q_T)$, $h = h(t_1, t_2, x) = t_1^s t_2^\ell |x|^r$, and is such that

$$\int_{Q_T} h|u|^p \zeta \, dP + \int_{S_T} \varphi_2(t_2, x)\zeta(0, t_2, x)\, dP_2 + \int_{S_T} \varphi_1(t_1, x)\zeta(t_1, 0, x)\, dP_1$$
$$= -\int_{Q_T} u\zeta_{t_1}\, dP - \int_{Q_T} u\zeta_{t_2}\, dP + \int_{Q_T} |u|^m (-\Delta)^{\alpha/2}\zeta \, dP, \tag{2.1}$$

for every regular function ζ, where $\zeta(T, t_2, x) = \zeta(t_1, T, x) = 0$, $P = (t_1, t_2, x)$, $P_1 = (t_1, x)$ and $P_2 = (t_2, x)$.

If in the definition $T = +\infty$, the solution is called global.