Computers and Mathematics with Applications 68 (2014) 2068–2079

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Eigenspectra of a complex coupled harmonic potential in three dimensions

S.B. Bhardwaj [a,b], Ram Mehar Singh [a,*], S.C. Mishra [c]

[a] Department of Physics, Ch. Devi Lal University, Sirsa-125055, India
[b] Department of Physics, Jai Parkash Mukand Lal Innovative Engineering and Technology Institute, Radaur (Haryana), India
[c] Department of Physics, Kurukshetra University, Kurukshetra-136119, India

A R T I C L E   I N F O

Article history:
Received 30 July 2013
Received in revised form 17 July 2014
Accepted 8 September 2014
Available online 23 October 2014

Keywords:
Ansatz
Complex potential
Eigenvalues and eigenfunctions

A B S T R A C T

Within the framework of extended complex phase space approach characterized by position and momentum coordinates, we investigate the quasi-exact solutions of the Schrödinger equation for a coupled harmonic potential and its variants in three dimensions. For this purpose ansatz method is employed and nature of the eigenvalues and eigenfunctions is determined by the analyticity property of the eigenfunctions alone. The energy eigenvalue is real for the real coupling parameters and becomes complex if the coupling parameters are complex. However, in case of complex coupling parameters, the imaginary component of energy eigenvalue reduces to zero if the $\mathcal{PT}$-symmetric condition is satisfied. Thus a non-hermitian Hamiltonian possesses real eigenvalue if it is $\mathcal{PT}$-symmetric.

1. Introduction

In recent years, the study of complex potentials has become more desirable [1–3] for the better theoretical understanding of several newly discovered phenomena in different contexts like optical model of a nucleus, delocalization transitions in condensed matter systems such as a vortex flux line depinning in type-II superconductors, population biology, Bose system of hard spheres, energy spectra of complex Toda lattice, quantum cosmology, quantum field theory and super symmetric quantum mechanics etc. [4–9]. As far as viability of the complex Hamiltonian [10,11] is concerned, it has been investigated in the context of quantum mechanics and semiclassical field theories. Complex Hamiltonians have also been studied in several other theoretical contexts—for example, the study of complex trajectories with regard to the calculation of a semiclassical coherent state propagator in the path integral method has attracted particular interest in the laser physics [11]. Besides some general studies of a complex Hamiltonian in a non-linear domain [1,12], some efforts have been made to study both classical [13,14] as well as quantum aspects [15,16] of a system. At the classical level, construction of exact invariants has been carried out using a more general transformation and the analyticity property of the Hamiltonian leads to a class of integrable systems. However, in the present study, we make use of such transformation to study the quantum aspect of a system and look for the analytic solutions of the Schrödinger equation (SE). It is to be noted that a complex Hamiltonian is no longer hermitian and ordinarily does not guarantee for real eigenvalues. However, in $\mathcal{PT}$-symmetric form [3,4,17], the system is found to exhibit real eigenvalue spectrum [18]. The reality of the spectrum is a direct consequence of combined action of parity and time reversal invariance of Hamiltonian [13,14]. The parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) operators defined

* Corresponding author.
E-mail address: dixit_rammehar@yahoo.co.in (R.M. Singh).

http://dx.doi.org/10.1016/j.camwa.2014.09.006
0898-1221/

by the action of position and momentum operators are

$$\hat{\mathcal{P}} : (x, y, z, p_x, p_y, p_z, i) \to (-x, -y, -z, -p_x, -p_y, -p_z, i),$$
$$\hat{\mathcal{T}} : (x, y, z, p_x, p_y, p_z, i) \to (x, y, z, -p_x, -p_y, -p_z, -i),$$

whereas, the combined action of $\hat{\mathcal{P}}\hat{\mathcal{T}}$-operation is given by

$$\hat{\mathcal{P}}\hat{\mathcal{T}} : (x, y, z, p_x, p_y, p_z, i) \to (-x, -y, -z, p_x, p_y, p_z, -i) \tag{1}$$

where $\hat{\mathcal{P}}^2\hat{\mathcal{T}}^2 = 1$. Such studies reveal that a complex Hamiltonian can give real and bounded energy eigenvalues for certain domain of underlying parameters. Therefore, it is possible to investigate the number of new non-hermitian Hamiltonian systems satisfying the $\mathcal{PT}$-symmetric condition. The type of hermiticity discussed here is different from that of given by Bender et al. [2–4]. But both approaches deal with different types of non-hermitian Hamiltonians, the non-hermiticity ($\mathcal{PT}$-symmetry) arising in the Bender's approach is due to the complexity of the potential parameters whereas in the present study, in addition to the potential parameters, the underlying phase space is also taken as complex. Thus $\mathcal{PT}$-symmetry discussed here is of generalized nature, which in certain limits reduces to the conventional $\mathcal{PT}$-symmetry [2,13].

There are various methods available in the literature for complexifying a given Hamiltonian system [14,15], but here we use the scheme given by Xavier and de-Aguiar [11] to transform the potential in extended complex phase space approach (ECPSA) characterized by

$$\begin{aligned} &x = x_1 + ip_4, \qquad y = x_2 + ip_5, \qquad z = x_3 + ip_6 \\ &p_x = p_1 + ix_4, \qquad p_y = p_2 + ix_5, \qquad p_z = p_3 + ix_6. \end{aligned} \tag{2}$$

The presence of variables $(x_4, x_5, x_6, p_1, p_2, p_3)$ in the above transformation represents coordinate-momentum interactions of a dynamical system. Note that in this complexifying scheme, the degrees of freedom of the underlying system just become double and the variables (x_1, p_1), (x_2, p_2), (x_3, p_3), (x_4, p_4), (x_5, p_5), (x_6, p_6) turn to be canonical pairs. Similar transformations (2) have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma [12].

In the past, considerable progress has been made on the front of complex Hamiltonian up to two-dimensional systems [14,19–21] but the same effort has not been acceded for three-dimensional coupled systems. However, some attempts have been made to obtain the solution of the SE for a coupled harmonic potential in real domain [20–23] but no step has been taken towards complex domain in higher dimensions. The study of complex coupled harmonic potential has become of considerable interest due to the peculiar nature of its eigenvalue spectrum. With this motivation and for gaining the further insight into the nature of the eigenspectra of a three-dimensional complex coupled harmonic potential, we investigate the quasi-exact solutions of the SE in an extended complex phase space.

The paper is structured as follows: in the next section, we demonstrate the ansatz method to enable its use in the subsequent sections. Using the same mathematical prescriptions, the explicit expressions for the energy eigenvalues and eigenfunctions of a coupled harmonic potential and its variant are presented in Section 3. Finally, concluding remarks are discussed in Section 4.

2. Ansatz method

The SE (for $\hbar = m = 1$) for a three-dimensional system is written as

$$\hat{H}(x, y, z, p_x, p_y, p_z)\psi(x, y, z) = E\psi(x, y, z), \tag{3}$$

where

$$\hat{H}(x, y, z, p_x, p_y, p_z) = -\frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) + V(x, y, z). \tag{4}$$

Transformation (2), implies that [20]

$$\begin{aligned} &\frac{d}{dx} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_4}\right), \qquad \frac{d}{dp_x} = \frac{1}{2}\left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_4}\right), \\ &\frac{d}{dy} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_5}\right), \qquad \frac{d}{dp_y} = \frac{1}{2}\left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_5}\right), \\ &\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial p_6}\right), \qquad \frac{d}{dp_z} = \frac{1}{2}\left(\frac{\partial}{\partial p_3} - i\frac{\partial}{\partial x_6}\right). \end{aligned} \tag{5}$$

On expressing $V(x, y, z)$, $\psi(x, y, z)$ and E in terms of real and imaginary components, we have

$$\begin{aligned} &V(x, y, z) = V_r(x_1, p_4, x_2, p_5, x_3, p_6) + iV_i(x_1, p_4, x_2, p_5, x_3, p_6), \\ &\psi(x, y, z) = \psi_r(x_1, p_4, x_2, p_5, x_3, p_6) + i\psi_i(x_1, p_4, x_2, p_5, x_3, p_6) \\ &E = E_r + iE_i, \end{aligned} \tag{6}$$