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# A multi-step differential transform method and application to non-chaotic or chaotic systems

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ABSTRACT

The differential transform method (DTM) is an analytical and numerical method for solving a wide variety of differential equations and usually gets the solution in a series form. In this paper, we propose a reliable new algorithm of DTM, namely multi-step DTM, which will increase the interval of convergence for the series solution. The multi-step DTM is treated as an algorithm in a sequence of intervals for finding accurate approximate solutions for systems of differential equations. This new algorithm is applied to Lotka–Volterra, Chen and Lorenz systems. Then, a comparative study between the new algorithm, multi-step DTM, classical DTM and the classical Runge–Kutta method is presented. The results demonstrate reliability and efficiency of the algorithm developed.



## 1. Introduction

The differential transform method (DTM) is a numerical as well as analytical method for solving integral equations, ordinary and partial differential equations. The method provides the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou [1] and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the $n$th derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in [2–23].

The DTM introduces a promising approach for many applications in various domains of science. However, DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. It is the purpose of this paper is to propose a reliable algorithm of the DTM. The new algorithm, multi-step DTM, presented in this paper, accelerates the convergence of the series solution over a large region and improve the accuracy of the DTM. The validity of the modified technique is verified through illustrative examples of Lotka–Volterra, Chen and Lorenz systems.

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**Table 1**
Operations of differential transformation.

| Original function | Transformed function |
|---|---|
| $f(t) = u(t) \pm v(t)$ | $F(k) = U(k) \pm V(k)$ |
| $f(t) = \alpha u(t)$ | $F(k) = \alpha U(k)$ |
| $f(t) = u(t)v(t)$ | $F(k) = \sum_{l=0}^{k} V(l)U(k-l)$ |
| $f(t) = \frac{du(t)}{dt}$ | $F(k) = (k+1)U(k+1)$ |
| $f(t) = \frac{d^m u(t)}{dt^m}$ | $F(k) = (k+1)(k+2)\cdots(k+m)U(k+m)$ |
| $f(t) = \int_{t_0}^{t} u(x)dx$ | $F(k) = \frac{U(k-1)}{k}, k \geq 1$ |
| $f(t) = t^m$ | $F(k) = \delta(k-m)$ |
| $f(t) = \exp(\lambda t)$ | $F(k) = \frac{\lambda^k}{k!}$ |
| $f(t) = \sin(wt+\alpha)$ | $F(k) = \frac{w^k}{k!}\sin(\pi k/2 + \alpha)$ |
| $f(t) = \cos(wt+\alpha)$ | $F(k) = \frac{w^k}{k!}\cos(\pi k/2 + \alpha)$ |

## 2. Differential transform method

The differential transform technique is one of the semi-numerical analytical methods for ordinary and partial differential equations that uses the form of polynomials as approximations of the exact solutions that are sufficiently differentiable. The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [2–5]. For convenience of the reader, we present a review of the DTM. The differential transform of the $k$th derivative of function $f(t)$ is defined as follows,

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0}, \tag{1}$$

where $f(t)$ is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is defined as,

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k. \tag{2}$$

From Eqs. (1) and (2), we get,

$$f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k f(t)}{dt^k}\bigg|_{t=t_0}, \tag{3}$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. For implementation purposes, the function $f(t)$ is expressed by a finite series and Eq. (2) can be written as,

$$f(t) \approx \sum_{k=0}^{N} F(k)(t-t_0)^k, \tag{4}$$

here $N$ is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1. The main steps of the DTM, as a tool for solving different classes of nonlinear problems, are the following. First, we apply the differential transform (1) to the given problem (integral equation, ordinary differential equation or partial differential equations), then the result is a recurrence relation. Second, solving this relation and using the differential inverse transform (2) we can obtain the solution of the problem.

## 3. Multi-step differential transform method

Although the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a very small region and it has slow convergent rate or completely divergent in the wider region [5–8]. To overcome the shortcoming, we present in this section the multi-step DTM that we have developed for the numerical solution of differential equations. For this purpose, we consider the following nonlinear initial value problem,

$$f(t, u, u', \ldots, u^{(p)}) = 0, \tag{5}$$

subject to the initial conditions $u^{(k)}(0) = c_k$, for $k = 0, 1, \ldots, p-1$.