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A dual Petrov–Galerkin finite element method for the convection–diffusion equation

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ABSTRACT

We present a minimum-residual finite element method (based on a dual Petrov–Galerkin formulation) for convection–diffusion problems in a higher order, adaptive, continuous Galerkin setting. The method borrows concepts from both the Discontinuous Petrov–Galerkin (DPG) method by Demkowicz and Gopalakrishnan (2011) and the method of variational stabilization by Cohen, Dahmen, and Welper (2012), and it can also be interpreted as a variational multiscale method in which the fine-scales are defined through a dual-orthogonality condition. A key ingredient in the method is the proper choice of dual norm used to measure the residual, and we present two choices which are observed to be robust in both convection and diffusion-dominated regimes, as well as a proof of stability for quasi-uniform meshes and a method for the weak imposition of boundary conditions. Numerically obtained convergence rates in 2D are reported, and benchmark numerical examples are given to illustrate the behavior of the method.

1. Introduction

It is well known that the standard Galerkin finite element method performs very poorly for the convection–diffusion equation—in the convection-dominated case, it experiences spurious oscillations in the solution that grow as $\epsilon \to 0$. The problem is connected back to a loss of discrete coercivity with respect to the standard H^1 norm [1]. The concept of *stabilized* methods was introduced in order to combat such oscillations, the most popular of which is the Streamline Upwind Petrov–Galerkin (SUPG) method [2]. The method can be interpreted as adding a sufficient amount of artificial viscosity in the streamline direction in order to restore discrete coercivity with respect to a new "streamline-diffusion" norm [3]. SUPG is also an example of a residual-based stabilization, where the stabilization mechanism disappears as the strong residual of the equation is satisfied.

A connection can be drawn between residual-based stabilized methods and Petrov–Galerkin schemes, where the trial (approximating) functions and test (weighting) functions are allowed to differ. Specifically, the SUPG method can be interpreted as a modification of standard test functions,[1] biasing them in the upwind direction based on mesh size, order of polynomial approximation, the magnitude of convection, and the diffusion parameter. More recently, the concept of

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[1] We note that one cannot recover the SUPG formulation beginning with the strong form of the equations and the SUPG test functions; the interpretation of SUPG as a Petrov–Galerkin scheme holds only locally, on element interiors.

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Petrov–Galerkin methods has been connected to a novel *minimum residual* framework—it is shown that the minimization of a specific residual corresponding to a variational formulation naturally leads to the concept of optimal test functions [4]. Additionally, this framework has been exploited as an alternative method of proving stability for more standard methods in the least squares finite element community [5].

Optimal test functions resulting from residual minimization were first implemented by Demkowicz and Gopalakrishnan in [6,7]. The connection between stabilization and least squares/minimum residual methods has been observed previously [8]; however, the Discontinuous Petrov–Galerkin method distinguishes itself by measuring the residual of the operator form of the equation, which is posed in the dual space. Independently, Cohen, Dahmen and Welper introduced an alternative saddle-point formulation of such a minimum residual method in [9], which alluded to a variational multiscale (VMS) perspective [10–12]. We refer to the method of Cohen, Dahmen, and Welper as a Dual Petrov–Galerkin method, which distinguishes itself from the Discontinuous Petrov–Galerkin method in that it does not use a broken test space.

The goal of this paper is three-fold. The first is to present a method which borrows concepts from each of these recent works and to derive a more detailed connection between the Petrov–Galerkin, stabilized, and VMS perspectives of minimum-residual methods. The second is to demonstrate that the proposed method is robust in both convection and diffusion-dominated regimes provided the dual norm used to measure the residual is chosen intelligently. The last goal is to show that the method is stable for arbitrary higher order and adaptive meshes and easily implemented using existing finite element codes and technologies.

2. Dual minimum-residual methods

Our starting point is the minimization of some measure of error over a finite-dimensional space. We begin by first introducing an abstract variational formulation

$$\begin{cases} \text{Given } l \in V^*, \quad \text{find } u \in U \text{ such that} \\ b(u, v) = l(v), \quad \forall v \in V, \end{cases} \tag{1}$$

where $b(\cdot, \cdot) : U \times V \to \mathbb{R}$ is a continuous bilinear form. Throughout the paper, we assume that the trial space U and test space V are real Hilbert spaces, and denote U^* and V^* as the respective topological dual spaces. Supposing the variational problem (1) to be well-posed in the inf–sup sense, we can then identify a unique operator $B : U \to V^*$ such that

$$\langle Bu, v\rangle_{V^*\times V} := b(u, v), \quad u \in U, v \in V$$

with $\langle \cdot, \cdot\rangle_{V^*\times V}$ denoting the duality pairing between V^* and V, to obtain the operator form of the variational problem

$$Bu = l \quad \text{in } V^*. \tag{2}$$

We define the residual $J(w)$ for $w \in U_h$ as

$$J(w) := \frac{1}{2}\|l - Bw\|_{V^*}^2 := \frac{1}{2}\sup_{v\in V\setminus\{0\}} \frac{|l(v) - b(w, v)|^2}{\|v\|_V^2}.$$

We are interested in determining u_h which minimizes the residual over the discrete approximating subspace $U_h \subset U$

$$u_h = \arg\min_{u_h\in U_h} J(u_h).$$

For convenience in writing, we will abuse the notation $\sup_{v\in V}$ to denote $\sup_{v\in V\setminus\{0\}}$ for the remainder of the paper. If we define the problem-dependent *energy norm*

$$\|u\|_E := \|Bu\|_{V^*},$$

then we can equate the minimization of $J(u_h)$ with the minimization of error in $\|\cdot\|_E$.

The first order optimality condition for minimization of the quadratic functional $J(u_h)$ requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$,

$$(l - Bu_h, B\delta u)_{V^*} = 0, \quad \forall \delta u \in U, \tag{3}$$

which is nothing more than the least-squares condition enforcing orthogonality of error with respect to the inner product on V^*.

The difficulty in working with the first-order optimality condition (3) is that the inner product $(\cdot, \cdot)_{V^*}$ cannot be evaluated explicitly. However, we have that

$$(l - Bu_h, B\delta u)_{V^*} = \left(R_V^{-1}(l - Bu_h), R_V^{-1}B\delta u\right)_V = 0, \tag{4}$$

where $R_V : V \to V^*$ is the Riesz map mapping elements of a Hilbert space V to elements of the dual V^* defined by

$$\langle R_V v, \delta v\rangle_{V^*\times V} := (v, \delta v)_V.$$

Furthermore, the Riesz operator is an isometry, such that $J(u_h) = \frac{1}{2}\|l - Bu_h\|_{V^*}^2 = \frac{1}{2}\left\|R_V^{-1}(l - Bu_h)\right\|_V^2$. Thus, satisfaction of (4) is exactly equivalent to satisfaction of the original optimality conditions (3).