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A robust Petrov–Galerkin discretisation of convection–diffusion equations☆

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ABSTRACT

A Petrov–Galerkin discretisation is studied of an ultra-weak variational formulation of the convection–diffusion equation in a mixed form. To arrive at an implementable method, the truly optimal test space has to be replaced by its projection onto a finite dimensional test search space. To prevent that this latter space has to be taken increasingly large for vanishing diffusion, a formulation is constructed that is well-posed in the limit case of a pure transport problem. Numerical experiments show approximations that are very close to the best approximations to the solution from the trial space, uniformly in the size of the diffusion term.

1. Introduction

It is well-known that standard Galerkin discretisations of convection–diffusion equations fail to deliver good approximations for a vanishing diffusion term. In this paper, we study *Petrov–Galerkin* discretisations.

Unless the layers are resolved by the mesh, the H^1-errors of finite element approximations will be dominated by the errors in the layers. This holds also true for L_2-errors when conforming finite elements are applied due to the strong enforcement of Dirichlet boundary conditions. Therefore, we prefer to measure the errors in the L_2-norm and to allow for discontinuous approximations. To this end, we consider an *ultra-weak* variational formulation of the convection–diffusion equation in a mixed form. It is shown to define a boundedly invertible mapping $U \to V'$, with U and V being Hilbert spaces, where U is (essentially) a multiple copy of the L_2-space.

Building on the earlier works [1–3], we equip V with the operator-dependent *optimal test norm*. Then given a finite dimensional trial space $U^h \subset U$, the Petrov–Galerkin discretisation with the *optimal test space* delivers the best approximation from U^h to the solution w.r.t. the norm on U.

To arrive at an implementable method, this truly optimal test space has to be replaced by its projection onto a finite dimensional *test search space*. With common variational formulations, the truly optimal test functions exhibit layers, and for vanishing diffusion, the test search space has to be chosen increasingly large to get satisfactory results.

In this paper, a non-standard variational formulation is constructed, such that for a zero diffusion term, the discrete system is a well-posed Petrov–Galerkin discretisation of the limiting transport problem. This can be seen as a necessary condition for the equations, which define the optimal test functions, not to be singularly perturbed.

Numerical experiments show that with a fixed test search space, the obtained approximations are very close to the *best* approximations to the solution from the trial space, uniformly in the size of the diffusion term.

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This paper is organised as follows. In Section 2, we revisit the theory of Petrov–Galerkin discretisations with optimal test spaces. In Section 3, we apply it to convection–diffusion equations, and in Section 4 we present numerical results.

In this work, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Some general theory

2.1. Petrov–Galerkin discretisations with optimal test spaces

For Hilbert spaces U and V over the scalar field $\mathbb{R}$, a bilinear form $b : U \times V \to \mathbb{R}$, let $(Bu)(v) := b(u, v)$ define a boundedly invertible mapping, i.e.,

$$B \in \mathcal{L}(U, V'), \quad B^{-1} \in \mathcal{L}(V', U). \tag{2.1}$$

Given $f \in V'$, we are interested in solving

$$Bu = f.$$

For defining our method, we will make use of $T \in \mathcal{L}(U, V)$ defined by

$$\langle Tu, v\rangle_V = b(u, v) \quad (u \in U,\ v \in V). \tag{2.2}$$

With the Riesz map $R_V \in \mathcal{L}(V, V')$ defined by $(R_V v)(z) = \langle v, w\rangle_V$ $(v, z \in V)$, it holds that $T = R_V^{-1}B$. Following [2], given a closed linear *trial space* $U^h \subset U$, we set the *optimal test space*

$$\operatorname{ran} T|_{U^h},$$

and consider the *Petrov–Galerkin* problem of finding $u^h \in U^h$ such that

$$b(u^h, v^h) = f(v^h) \quad (v^h \in \operatorname{ran} T|_{U^h}). \tag{2.3}$$

As will follow as a special case from Proposition 2.2, (2.3) has a unique solution, and it holds that

$$u^h = \operatorname*{argmin}_{\bar{u}^h \in U^h} \|f - B\bar{u}^h\|_{V'},$$

so that actually the Petrov–Galerkin discretisation with optimal test space is a least-squares method.

Only in cases where the dual norm $\|\cdot\|_{V'}$ can be evaluated exactly, this least-squares problem can be solved exactly. For this reason, in the following subsection we consider Petrov–Galerkin discretisations with *projected optimal test spaces*.

2.2. Petrov–Galerkin with projected optimal test spaces

Given a closed linear trial space $U^h \subset U$, let $V^h \subset V$ be a sufficiently large closed subspace, that we call *test search space*, such that

$$\gamma^h := \inf_{0 \neq w^h \in U^h} \sup_{0 \neq v^h \in V^h} \frac{b(w^h, v^h)}{\|w^h\|_U \|v^h\|_V} > 0. \tag{2.4}$$

Thanks to (2.1), in any case the latter is satisfied for $V^h = V$, with $\gamma^h \geq \|B^{-1}\|_{V' \to U}^{-1}$ (with equality when $U^h = U$).

Remark 2.1 (*Fortin Projector*)**.** From [4], we recall that if there exists a projector $\Pi^h \in \mathcal{L}(V, V^h)$ with $b(w^h, \Pi^h v) = b(w^h, v)$ $(w^h \in U^h)$, then

$$\gamma^h \geq \inf_{0 \neq w^h \in U^h} \sup_{0 \neq v \in V} \frac{b(w^h, \Pi^h v)}{\|w^h\|_U \|\Pi^h v\|_V} \geq \frac{1}{\|\Pi^h\|_{V \to V} \|B^{-1}\|_{V' \to U}}.$$

Conversely, if (2.4) is valid, then defining $\Pi^h v$ as the first component of the solution $(v^h, \lambda^h) \in V^h \times U^h$ of

$$\langle v^h, z^h\rangle_V + b(\lambda^h, z^h) = \langle v, z^h\rangle_V \quad (z^h \in V^h),$$
$$b(w^h, v^h) = b(w^h, v) \quad (w^h \in U^h),$$

a projector as above is constructed, with $\|\Pi^h\|_{V \to V} \lesssim (\gamma^h)^{-1}$.

We define $T^h \in \mathcal{L}(U, V^h)$ by

$$\langle T^h u, v^h\rangle_V = b(u, v^h) \quad (u \in U,\ v \in V^h), \tag{2.5}$$

whose existence is guaranteed by Riesz' representation theorem.

Given a closed linear trial space $U^h \subset U$, we set the *projected optimal test space* by

$$\operatorname{ran} T^h|_{U^h},$$