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Linear and quasi-linear spaces of set-valued maps

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ABSTRACT

In this paper we investigate the algebraic structure of certain spaces of set-valued maps. Among other results, we show that for an arbitrary topological space X and a metrisable topological vector space Z, the space $\mathcal{M}(X, Z)$ of minimal upper semi-continuous compact valued (musco) maps from X into Z is a linear space. This result extends a previously known result on the linear structure of spaces of musco maps. Previously, this result was known only in the case when X is a Baire space. We also study topologies of uniform convergence on compact sets on $\mathcal{M}(X, Z)$.

1. Introduction

Set-valued maps appear in natural ways in various parts of mathematics and its applications to the life sciences and economics, among others, see for instance [1–5]. Here we may mention that biological dynamic systems typically involve uncertain data and/or parameters, numerical and/or inherent sensitivity and structural uncertainties which necessitate model validation. Problems related to these issues of uncertainty and sensitivity essentially belong to Set-Valued Analysis.

The history of the subject dates back at least to the nineteenth century when set-valued maps appeared in connection with the theory analytic functions of a complex variable developed mainly by Cauchy, Riemann and Weierstrass. Despite its long history and natural role in many mathematical problems, the study of set-valued maps in the functional analytic context, where the emphasis is placed on spaces of functions and functions defined on such spaces, has until recently received little attention. Here we may mention the early work of Aseev [6] and the more recent contributions of Anguelov and Kalenda [7], Hammer and McCoy [8], McCoy [9,10] and Holá [11–13]. The current paper is a contribution to this direction of inquiry.

We continue our investigation [14] of the algebraic structure of spaces of set-valued maps. The main result in this regard states that the set of minimal upper semi-continuous compact valued maps from an arbitrary topological space X into a metrisable topological vector space Z is a vector space. This generalises a result in [14] where X is required to be a Baire space. The definition of the algebraic operations in [14] relies in an essential way on the fact that every minimal upper semi-continuous compact valued map from a Baire space X into a metric Y space is point-valued on a dense G-δ set. Since this is not true for a general topological space X, a new technique must be developed for the general case considered here. The key to our result is a metric characterisation of quasi-minimal upper semi-continuous compact valued maps. We also consider a variant the topology of uniform convergence on compact subsets of X introduced in [8]. This leads to generalisations of some results of McCoy [10] and Holá [12].

The paper is organised as follows. In Section 2, for the convenience of the reader, we recall some basic definitions and results concerning set-valued maps that will be referred to in subsequent sections. Section 3 deals with the algebraic structure of spaces of set-valued maps, as described in the previous paragraph, while Section 4 is concerned with the topology of (strong) uniform convergence on compact sets.

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Finally, let us fix some notation. Unless specified otherwise, X and Y will denote general topological spaces, while Z is a topological vector space over a field $\mathbb{K}$, with $\mathbb{K}$ being either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. For every $x \in X$, $\mathcal{V}_x$ denotes the set of open neighbourhoods of x in X. The closure of $A \subseteq X$ is denoted by $\overline{A}$, while $\mathrm{Int}(A)$ is the interior of A. If Y is a metric space, we denote by $B(y, \epsilon)$ the open ball centred at $y \in Y$ with radius $\epsilon > 0$, while $\overline{B}(y, \epsilon)$ is the closure of $B(y, \epsilon)$. If it is necessary to emphasise the particular metric d on Y, the ball centred at $y \in Y$ with radius ϵ is denoted by $B_d(y, \epsilon)$. The set of all subsets of Y is denoted by $\mathcal{P}(Y)$, while $\mathcal{C}(Y)$ consists of all nonempty, closed subsets of Y. The set of all nonempty compact subsets of Y is denoted by $\mathcal{K}(Y)$.

2. Set-valued maps

A set-valued map $f : X \rightrightarrows Y$ from X into Y is a function $f : X \to \mathcal{P}(Y)$. That is, $f(x)$ is a subset of Y for every $x \in X$. We consider only set-valued maps with nonempty values, that is, we assume that $f(x) \neq \emptyset$ for all $x \in X$. By identifying a point $y \in Y$ with the singleton $\{y\} \in \mathcal{P}(Y)$, we see that every function $f : X \to Y$ corresponds in a natural way to a set-valued mapping, namely, the mapping $X \ni x \mapsto \{f(x)\} \in \mathcal{P}(Y)$. We will not distinguish between the point-valued function f and the associated set-valued map, and will denote them by the same symbol.

By identifying a set-valued map $f : X \rightrightarrows Y$ with its graph

$$G(f) = \{(x, y) \in X \times Y \ : \ y \in f(x)\},$$

we may define the set-theoretic notions of inclusion, intersection and union for such maps. In particular, for $f, g : X \rightrightarrows Y$ we say that $f \subseteq g$ whenever $G(f) \subseteq G(g)$, that is, $f \subseteq g$ if and only if $f(x) \subseteq g(x)$ for every $x \in X$. Likewise, the union of f and g is the map $f \cup g : X \rightrightarrows Y$ with graph $G(f \cup g) = G(f) \cup G(g)$ so that $f \cup g(x) = f(x) \cup g(x), \ x \in X$, while $f \cap g(x) = f(x) \cap g(x)$ provided the intersection is non-empty.

Next we recall some notions of continuity of set-valued maps.

Definition 1. $f : X \rightrightarrows Y$ is upper semi-continuous at $x_0 \in X$ if for every open set $U \supseteq f(x_0)$ there exists $V \in \mathcal{V}_{x_0}$ so that $f(x) \subseteq U$ for every $x \in V$. If f is upper semi-continuous at every $x \in X$, then we say that f is upper semi-continuous.

Definition 2. $f : X \rightrightarrows Y$ is lower semi-continuous at $x_0 \in X$ if for every open set $U \subseteq Y$ so that $f(x_0) \cap U \neq \emptyset$ there exists $V \in \mathcal{V}_{x_0}$ so that $f(x) \cap U \neq \emptyset$ for every $x \in V$. If f is lower semi-continuous at every $x \in X$, then we say that f is lower semi-continuous.

Definition 3. $f : X \rightrightarrows Y$ is continuous at $x \in X$ if f is both upper semi-continuous and lower semi-continuous at $x \in X$. If f is continuous at every $x \in X$, then f is said to be continuous.

If $f : X \rightrightarrows Y$ is upper semi-continuous and compact valued, that is, $f(x) \in \mathcal{K}(Y)$ for every $x \in X$, we say that f is *usco*. An usco map f is called *minimal (musco)* if $f = g$ for all usco maps $g \subseteq f$, while f is called quasi-minimal if it contains exactly one musco map. For a quasi-minimal usco map f we denote by $\langle f \rangle$ the unique musco map contained in f. The set of all musco maps is denoted by $\mathcal{M}(X, Y)$, while $\mathcal{Q}(X, Y)$ is the set of all quasi-minimal usco maps. The surjective map

$$\langle \cdot \rangle : \mathcal{Q}(X, Y) \ni f \mapsto \langle f \rangle \in \mathcal{M}(X, Y) \qquad (1)$$

will play an important role in the constructions presented in Section 3.

Clearly every continuous function $f : X \to Y$ is usco, and in fact musco. Therefore the inclusions

$$\mathcal{C}(X, Y) \subseteq \mathcal{M}(X, Y) \subseteq \mathcal{Q}(X, Y)$$

hold. In general, equality fails for both inclusions as can readily be seen at the hand of elementary examples. However, every musco map is nearly everywhere continuous, see [15].

Proposition 1. *Assume that X is a Baire space and Y is a metric space. If $f : X \rightrightarrows Y$ is usco, then there exists a dense $G - \delta$ set $D \subseteq X$ so that f is continuous at every point in D. Furthermore, if f is musco, then f is point valued at every point in D.*

Let us now recall those results dealing with usco maps that will be used in subsequent sections. Since, in this paper, we are concerned with compact valued maps only, all results in this section will be formulated for compact valued maps, even in cases when the result is known in a more general form. Firstly, musco maps are abundant, at least relative to the usco maps, as is shown by the following result, see for instance [16].

Proposition 2. *If $f : X \rightrightarrows Y$ is usco, then there exists a musco map $g : X \rightrightarrows Y$ so that $g \subseteq f$.*

New usco maps may be generated from a given usco map, or family of usco maps, in the following ways. These results can be found in [17], among others, except for Proposition 5 which is due to Borwein and Moors, see [18].

Proposition 3. *The following statements are true for all usco maps $f, g : X \rightrightarrows Y$.*

(i) *$f \cup g$ is usco.*
(ii) *If $f(x) \cap g(x) \neq \emptyset$ for every $x \in X$, then $f \cap g$ is usco.*