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A coupled anisotropic chemotaxis-fluid model: The case of two-sidedly degenerate diffusion

Georges Chamoun [a,b,*], Mazen Saad [a], Raafat Talhouk [b]

[a] Ecole Centrale de Nantes, Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629, 1 rue de la Noé, 44321 Nantes, France

[b] Université Libanaise, EDST et Faculté des sciences I, Laboratoire de Mathématiques, Hadath, Beyrouth, Liban

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ABSTRACT

In this article, the mathematical analysis of a model arising from biology consisting of diffusion, chemotaxis with volume filling effect and transport through an incompressible fluid, is studied. Motivated by numerical and modeling issues, the global-in-time existence of weak solutions to this model is investigated. The novelty with respect to other related papers lies in the presence of two-sidedly nonlinear degenerate diffusion and of anisotropic and heterogeneous diffusion tensors where we prove the global existence for a Chemotaxis-Navier–Stokes system in space dimensions less than or equal to four and we show the uniqueness of weak solutions for the Chemotaxis-Stokes system in two or three space dimensions under further assumptions.

1. Introduction

Chemotaxis is the movement of biological individuals towards (or away from) a chemoattractant (or chemorepellent). A vital characteristic of living organisms is the ability to sense signals in the environment and adapt their movement accordingly. This behavior enables them to locate nutrients, avoid predators or find animals of the same species. A typical model describing chemotaxis is the Keller–Segel equations derived by Keller and Segel [1] which have become one of the best-studied models in mathematical biology. In nature, cells often live in a viscous fluid so that cells and chemical substrates are also transported with the fluid, and meanwhile the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells. Thus, it is interesting and important in biology to study some phenomenon of chemotaxis on the basis of the coupled cell–fluid model. In the following, we investigate a system consisting of the parabolic chemotaxis equations with general tensors coupled to Navier–Stokes equations,

$$\begin{cases} \partial_t N - \nabla \cdot \big(S(x)a(N)\nabla N\big) + \nabla \cdot \big(S(x)\chi(N)\nabla C\big) + u \cdot \nabla N = f(N), \\ \partial_t C - \nabla \cdot (M(x)\nabla C) + u \cdot \nabla C = -k(C)N, \\ \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla P = -N\nabla\phi, \\ \nabla \cdot u = 0, \quad t > 0, x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$, $d \leq 4$ with smooth boundary $\partial\Omega$. The experimental set-up corresponds to mixed type boundary conditions. For simplicity here we use null flux conditions for N and C and zero Dirichlet for u to

* Corresponding author at: Ecole Centrale de Nantes, Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629, 1 rue de la Noé, 44321 Nantes, France.
E-mail addresses: Georges.Chamoun@ec-nantes.fr (G. Chamoun), Mazen.Saad@ec-nantes.fr (M. Saad), rtalhouk@ul.edu.lb (R. Talhouk).

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reflect the no-slip boundary conditions of the flow. Therefore, this system of equations is supplemented by the following boundary conditions on $\Sigma_t = \partial\Omega \times (0, T)$,

$$S(x)a(N)\nabla N \cdot \eta = 0, \qquad M(x)\nabla C \cdot \eta = 0, \qquad u = 0, \tag{1.2}$$

where η is the exterior unit normal to $\partial\Omega$. The initial conditions on Ω are given by,

$$N(x, 0) = N_0(x), \qquad C(x, 0) = C_0(x), \qquad u(x, 0) = u_0(x). \tag{1.3}$$

Here N, C, u and P denote respectively the cell density, the concentration of a chemical, the velocity field and the pressure inside the incompressible fluid. Moreover, $a(N)$ denotes the density-dependent diffusion coefficient and $\chi(N)$ is usually written in the form $\chi(N) = Nh(N)$ where h is commonly referred to as the chemotactic sensitivity function. The source term f reflects the interaction between cells such as hydrodynamics interactions. Anisotropic and heterogeneous tensors are denoted by $S(x)$ and $M(x)$. The fluid is described by an incompressible Navier–Stokes equation with the viscosity ν. It couples to N and C through transport by the fluid modeled by $u \cdot \nabla N$, $u \cdot \nabla C$ and gravitational forcing modeled by $g = -N\nabla\phi$ as an external force exerted on the fluid by the cells. In fact, this external force can be produced by different physical mechanisms such as gravity, electric and magnetic forces but in our study, we are only interested in the case of gravitational force $\nabla\phi = "V_b(\rho_b - \rho)g''z$ exerted by a bacterium onto the fluid along the upwards unit vector z proportional to the volume of the bacterium V_b, the gravitation acceleration $g = 9.8\ \mathrm{m/s^2}$, and the density of bacteria is ρ_b (bacteria are about 10% denser than water). Moreover, since the fluid is slow, we can use the Stokes equation instead of the Navier–Stokes equation. So the system looks like,

$$\begin{cases} \partial_t N - \nabla \cdot \big(S(x)a(N)\nabla N\big) + \nabla \cdot \big(S(x)\chi(N)\nabla C\big) + u \cdot \nabla N = f(N), \\ \partial_t C - \nabla \cdot (M(x)\nabla C) + u \cdot \nabla C = -k(C)N, \\ \partial_t u - \nu\Delta u + \nabla P = -N\nabla\phi, \\ \nabla \cdot u = 0 \quad t > 0, x \in \Omega. \end{cases} \tag{1.4}$$

In the models (1.1) and (1.4), the cell density N diffuses, it moves in the direction of the chemical gradient and it is transported by the fluid. In addition to that, the chemical C also diffuses, it is also transported by the fluid and it is consumed proportional to the density of cells N, where this fact is expressed by a function $k(C)$ which is a consumption rate of the chemical by the cells. In this paper, the chemical substrate can be only consumed by the cells ($\tilde{g}(N, C) = -k(C)N$). For example, the bacteria "*Bacillus subtilis*" swim towards higher concentration of oxygen to survive. In other cases, such as the "*Dictyostelium discoideum*", the chemical can be produced and consumed ($\tilde{g}(N, C) = aN - bC$ where a and b are positive constants) to form some kind of transition to a multicellular organism. The theoretical study of this paper is valid for both cases (chemotactical transport and transport towards a nutrient) even we are only considering the first one in the sequel. There are also an another possible choice of $\tilde{g}$ as a cut-off function for which many related experiments have been given in [2–4] to describe the aggregation of a part of bacteria below an interface between two fluids, while other bacteria are rendered inactive wherever the oxygen concentration has fallen below the threshold of activity.

Motivated by experiments described in [5,6] which explain the dynamics of anisotropic chemotaxis models in a fluid at rest ($u = 0$) and interested by numerical issues related to the dynamics of these models coupled to a viscous fluid through transport and gravitational force, we investigate in this paper the coupled anisotropic chemotaxis-fluid models (1.1) and (1.4). A detailed theoretical study of global existence and uniqueness of weak solutions of these models has been established. In fact, the existence theory in suitable functional spaces and the uniqueness can present several difficulties due to the complicated cell–fluid interaction even if it only consists of chemotaxis and linear isotropic non-degenerate diffusion coupled to the fluid. Indeed, in the case of isotropic homogeneous tensors ($S(x) = M(x) = Id$), linear diffusion ($a(N) = 1$) and a concentration-dependent sensitivity ($\chi(N, C) = N\beta(C)$ where $\beta(C)$ is the chemotactic sensitivity), several authors of the chemotaxis literature have recently studied the global existence in time via finite time blow-up of a weak solution for the models (1.1) and (1.4). The main tool used to prove global existence is an existing entropy inequality. In [7], the authors proved the global existence for the model (1.4) for weak potential ϕ or small initial data of the concentration C. Moreover, for $\Omega = \mathbb{R}^2$ or $\mathbb{R}^3$, by changing the consumption rate ($-k(C)N$) into a production one ($N - aC$ where $a > 0$) and by considering the stationary equation of C, the authors in [8] proved the existence of a critical initial mass M in the model (1.4), below M we have the global existence and above M we have finite time blow-up. For $\Omega = \mathbb{R}^2$, the global existence in time of a weak solution for the model (1.1) is proved in [8]. In addition to that, for the case of isotropic tensors, nonlinear diffusion ($a(N) = mN^{m-1}\nabla N$) which degenerates only at one point ($u = 0$) and for the same sensitivity ($\chi(N, C) = N\beta(C)$), the global existence of a weak solution for the model (1.4) is proved in [9] for $\Omega = \mathbb{R}^2$ and also proved for $\frac{4}{3} < m \leq 2$ where Ω is bounded in $\mathbb{R}^2$. Moreover, the case of $m = \frac{4}{3}$ in the whole space $\Omega = \mathbb{R}^3$ is treated also in [10]. To our knowledge, these are the only results on models related to (1.1) and (1.4).

The purpose of this paper is twofold: on the one hand, we establish the global-in-time existence of weak solutions to the models (1.1) and (1.4) in the open bounded domain Ω ($\Omega \subset \mathbb{R}^d$, $d \leq 4$), in the presence of anisotropic and heterogeneous tensors, two-sidedly nonlinear degenerate diffusion, modified chemotactic sensitivity χ and Navier–Stokes equations. On the other hand, we prove the uniqueness of weak solutions to the system (1.4) in Ω ($\Omega \subset \mathbb{R}^d$, $d = 2, 3$) under further assumptions and regularities on the initial data.