Computers and Mathematics with Applications 57 (2009) 1456–1462

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of solutions for a class of hemivariational inequality problems[☆]

Yisheng Huang[*], Yuying Zhou

Department of Mathematics, Suzhou University, Suzhou 215006, Jiangsu, PR China

ARTICLE INFO

Article history:
Received 11 February 2008
Received in revised form 15 December 2008
Accepted 9 January 2009

Keywords:
Hemivariational inequality
Generalized pseudomonotone mapping
The Clarke generalized gradient
Galerkin approximation
Eigenvalue

ABSTRACT

In this paper, we are concerned with the existence of solutions for a class of Hartman–Stampacchia type hemivariational inequalities by using the Clarke generalized directional derivative and the Galerkin approximation method. Two existence results of solutions for the generalized pseudomonotone mapping hemivariational inequality and elliptic hemivariational inequality are obtained.

1. Introduction

Let $\mathscr{K}$ be a subset of a real Banach space $\mathscr{X}$ and let $\mathscr{X}^*$ be the dual space of $\mathscr{X}$. Suppose that $A : \mathscr{K} \to \mathscr{X}^*$ is an operator and $T : \mathscr{X} \to L^p(\Omega; \mathbb{R}^m)$ is a linear continuous operator, where $1 \le p < \infty$, and Ω is a bounded open set in $\mathbb{R}^N$. For each $u \in \mathscr{X}$, we denote $\hat{u}$, an element of $L^p(\Omega; \mathbb{R}^m)$, by $\hat{u} := Tu$. Let $j : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that j is locally Lipschitz with respect to the second variable $y \in \mathbb{R}^m$. Suppose that j satisfies:

(J) there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega; \mathbb{R})$ and $h_2 \in L^\infty(\Omega; \mathbb{R})$ such that

$$|\zeta| \le h_1(x) + h_2(x)|y|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^m \text{ and } \zeta \in \partial j(x, y).$$

In this paper, we will be concerned with the existence of solutions of the following hemivariational inequality problem:

(P) Find $u \in \mathscr{K}$ such that, for every $v \in \mathscr{K}$,

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\mathrm{d}x \ge 0, \quad \forall v \in \mathscr{K},$$

where $\langle \cdot, \cdot\rangle$ means the duality pairing between $\mathscr{X}$ and $\mathscr{X}^*$, $j^0(x, y; h)$ denotes the Clarke generalized directional derivative of the locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^m$ with respect to the direction $h \in \mathbb{R}^m$ (where $x \in \Omega$), while $\partial j(x, y)$ is the Clarke generalized gradient of this mapping at $y \in \mathbb{R}^m$, that is,

$$j^0(x, y; h) = \limsup_{y' \to y,\ t \to 0^+} \frac{j(x, y' + tv) - j(x, y')}{t},$$

$$\partial j(x, y) = \{\zeta \in \mathbb{R}^m : \langle \zeta, h\rangle \le j^0(x, y; h),\ \text{for all } h \in \mathbb{R}^m\}$$

(one can see, e.g. [1], for the definitions of the generalized directional derivative and the generalized gradient of a locally Lipschitz functional in a Banach space).

When $j(x, y) \equiv 0$, $\forall (x, y) \in \Omega \times \mathbb{R}^m$, $\mathscr{X}$ is a finite-dimensional Banach space, $\mathscr{K}$ is a nonempty, compact and convex set of $\mathscr{X}$ and the operator A is continuous. The variational inequality problem (P) which was first proposed by Hartman and

☆ Supported by the Natural Science Foundation of China (No. 10571174) and a grant (No. 08KJB110009) from Jiangsu Education Committee of China.

* Corresponding author.
E-mail address: yishengh@suda.edu.cn (Y. Huang).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.01.026

Stampacchia in 1966 (see [2]) is well known. Hemivariational inequality problem is a new type of inequality problem which was introduced by Panagiotopoulos (cf. [3]) in order to deal with problems in mechanics and engineering whose variational forms are such inequalities which express the principle of virtual work or power. The variational and hemivariational inequalities have been investigated by a number of authors; the reader is referred to [4–11] and the references therein, where the treatment relies on monotonicity principles, projection arguments, topological method and nonsmooth critical point theory.

In this paper, we will get the existence results of solutions to the hemivariational inequality problem (P) of the Hartman–Stampacchia type by using a finite-dimensional approximation method which was used in [12] by authors to obtain the existence of solutions for an elliptic variational inequality. Namely, first we obtain an existence result for solutions of a hemivariational inequality with a continuous mapping satisfying the Karamandian condition in finite spaces; then by the Galerkin approximation, we obtain an existence result for the hemivariational inequality problem with a generalized pseudomonotone mapping in infinite spaces (this result is more general than that in [5, Theorem 3.10.4] and [7, Theorem 6.2]); finally, as an application, we use this result to establish a new existence result for an elliptic hemivariational inequality problem which was considered in [7,13,14]. Note that the assumptions and the methods given in this paper are different from those in [5,7,13,14].

2. Preliminaries

In this section, we will present some definitions and lemmas which we will use to get our main results.

Definition 2.1 (*[15]*). A mapping $A : \mathscr{K} \to \mathscr{X}^*$ is said to be generalized pseudomonotone if, $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{K}$ satisfies $u_n \rightharpoonup u$, $Au_n \rightharpoonup w$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$, we have $w = Au$ and $\langle Au_n, u_n\rangle \to \langle w, u\rangle$ as $n \to \infty$.

Definition 2.2 (*[16]*). Let $\mathscr{K}$ be a cone of the Banach space $\mathscr{X}$. A Galerkin approximation of the cone $\mathscr{K}$ is a countable family of cones $\{\mathscr{K}_n\}_{n\in\mathbb{N}}$ satisfying the following properties:

(1) $\mathscr{K}_n \subset \mathscr{K}\ \forall\, n \in \mathbb{N}$;
(2) $\dim(\mathscr{K}_n) < +\infty\ \forall\, n \in \mathbb{N}$;
(3) $\lim_{n\to\infty}(\mathrm{Proj}|_{\mathscr{K}_n} x) = x, \forall x \in \mathscr{K}$.

A cone $\mathscr{K}$ is called a *Galerkin cone* if $\mathscr{K}$ has a Galerkin approximation.

Definition 2.3. Let $\mathscr{K}$ be a cone of the Banach space $\mathscr{X}$, $j : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathédory function. A mapping $A : \mathscr{K} \to \mathscr{X}^*$ is said to satisfy the Karamandian condition with j if, there exists a compact convex subset $\mathscr{A} \subset \mathscr{K}$, such that for each $y \in \mathscr{K} \setminus \mathscr{A}$ there exists $z \in \mathscr{A}$ satisfying

$$\langle Ay, y - z\rangle > \int_\Omega j^0(x, \hat{y}(x); \hat{z}(x) - \hat{y}(x))\mathrm{d}x. \tag{1}$$

Remark 2.1. Definition 2.3 can be seen in [17] if $j(x, y) \equiv 0$ for all $(x, y) \in \Omega \times \mathbb{R}^m$.

Definition 2.4. Let $\{\mathscr{K}_n\}$ be a Galerkin approximation of a cone $\mathscr{K} \subset \mathscr{X}$. If there exists a family $\{\mathscr{D}_n\}$ of equibounded, closed convex sets such that $\mathscr{D}_n \subset \mathscr{K}_n$ ($\forall n \in \mathbb{N}$), and for each $y \in \mathscr{K}_n \setminus \mathscr{D}_n$ there exists $z \in \mathscr{D}_n$ satisfying (1), then the mapping A satisfies *the generalized Karamandian condition* with j on $\mathscr{K}$.

Lemma 2.1 (*[7, Lemma 6.1]*). *If j satisfies the assumption* (J) *and X_1, X_2 are nonempty subsets of $\mathscr{X}$, then the mapping $(u, v) \mapsto \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))\mathrm{d}x$ from $X_1 \times X_2$ to $\mathbb{R}$ is upper semicontinuous. Moreover, if $T : \mathscr{X} \to L^p(\Omega, \mathbb{R}^m)$ is a linear compact operator, then the above mapping is weakly upper semicontinuous.*

Lemma 2.2. *Suppose that $\mathscr{M}$ is a finite-dimensional Banach space, $\mathscr{K}_0$ is a cone of $\mathscr{M}$, the function $j : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies the assumption* (J), *mapping $A : \mathscr{K}_0 \to \mathscr{M}$ is continuous and satisfies the Karamandian condition with j. Then Problem* (P) *has a solution.*

Proof. Since A satisfies the Karamandian condition with j, there exists a compact convex subset $\mathscr{A} \subset \mathscr{K}_0$, such that for each $y \in \mathscr{K}_0 \setminus \mathscr{A}$ there exists $z \in \mathscr{A}$ satisfying

$$\langle Ay, y - z\rangle > \int_\Omega j^0(x, \hat{y}(x); \hat{z}(x) - \hat{y}(x))\mathrm{d}x. \tag{2}$$

Let $\{u_1, u_2, \ldots, u_m\} \subset \mathscr{K}_0$ and let $\mathscr{B}$ be the convex hull of $\mathscr{A} \cup \{u_1, u_2, \ldots, u_m\}$. Clearly, $\mathscr{B}$ is a compact convex subset of $\mathscr{K}_0$. By [7, Corollary 6.1], there exists $u_0 \in \mathscr{B}$ such that

$$\langle Au_0, v - u_0\rangle + \int_\Omega j^0(x, \hat{u_0}(x); \hat{v}(x) - \hat{u_0}(x))\mathrm{d}x \ge 0, \quad \forall v \in \mathscr{B}. \tag{3}$$