Computers and Mathematics with Applications 57 (2009) 1574–1582

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

New families of nonlinear third-order solvers for finding multiple roots

Changbum Chun [a], Hwa ju Bae [a], Beny Neta [b,*]

[a] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*

[b] *Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, United States*

ARTICLE INFO

Article history:
Received 16 September 2008
Accepted 16 October 2008

Keywords:
Newton's method
Multiple roots
Iterative methods
Nonlinear equations
Order of convergence
Root-finding

ABSTRACT

In this paper, we present two new families of iterative methods for multiple roots of nonlinear equations. One of the families require one-function and two-derivative evaluation per step, and the other family requires two-function and one-derivative evaluation. It is shown that both are third-order convergent for multiple roots. Numerical examples suggest that each family member can be competitive to other third-order methods and Newton's method for multiple roots. In fact the second family is even better than the first.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a multiple root α of multiplicity m, i.e., $f^{(j)}(\alpha) = 0,\ j = 0, 1, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

The well known Newton's method for finding a multiple root α is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

which converges quadratically [1].

There exists a cubically convergent Halley method [2] which Hansen and Patrick [3] extended to multiple roots, which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}. \tag{2}$$

In recent years, some modifications of the Newton method for multiple roots have been proposed and analyzed, see for example [4–14] and references therein. These methods have been proven to be competitive to Newton's method in their performance and efficiency. There are, however, not yet so many methods known in open literature that can handle the case of multiple roots, see [13]. To deal with the multiple roots case, one may use the observation that the functions

$$u = \frac{f}{f'}, \qquad f^{(m-1)}, \qquad f^{1/m}$$

* Corresponding author.
E-mail addresses: cbchun@skku.edu (C. Chun), bneta@nps.edu, byneta@gmail.com (B. Neta).

0898-1221/$ – see front matter. Published by Elsevier Ltd
doi:10.1016/j.camwa.2008.10.070

have only a simple zero at α, and any of the iterative methods for a simple zero may be used [15]. However, this approach might become problematic due to higher computational costs. This being the case, development of iterative methods to approximate a multiple root is required, and this is our motivation for this work.

In this paper, we present two new third-order families of methods for multiple roots. The first one is based on the composition of Osada's third-order multiple root-finding method [4]

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}, \tag{3}$$

and the Euler–Chebyshev third-order multiple root-finding method [15]

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f(x_n)}{f'(x_n)} - \frac{m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}. \tag{4}$$

This family proposed here is shown to be locally cubically convergent. Its performance is often a little better than the two third-order methods from which this family is derived, and its practical utility is demonstrated by numerical examples.

The other method is based on one of the third order methods due to Dong [5], i.e.

$$y_n = x_n - u_n, \tag{5}$$

$$x_{n+1} = y_n + \frac{u_n f(y_n)}{f(y_n) - (1-\frac{1}{m})^{m-1} f(x_n)}, \tag{6}$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)} \tag{7}$$

and a third order method due to Victory and Neta [6], i.e.

$$y_n = x_n - u_n, \tag{8}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}\frac{f(x_n) + Af(y_n)}{f(x_n) + Bf(y_n)}, \tag{9}$$

where

$$A = \mu^{2m} - \mu^{m+1}, \tag{10}$$

$$B = -\frac{\mu^m(m-2)(m-1)+1}{(m-1)^2}, \tag{11}$$

$$\mu = \frac{m}{m-1}. \tag{12}$$

This family is also of third order but requires two-function and one-derivative evaluations.

2. Development of methods and convergence analysis

To derive the first method, let us consider the composition of the methods (3) and (4) in the form

$$\begin{aligned} x_{n+1} &= x_n - \frac{\theta}{2}\left[m(m+1)\frac{f(x_n)}{f'(x_n)} - (m-1)^2\frac{f'(x_n)}{f''(x_n)}\right] \\ &\quad - \frac{(1-\theta)}{2}\left[m(3-m)\frac{f(x_n)}{f'(x_n)} + m^2\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}\right], \end{aligned} \tag{13}$$

where $\theta \in \mathbf{R}$, from which we suggest the following one-parameter family of methods for multiple roots

$$x_{n+1} = x_n - \frac{m[(2\theta-1)m+3-2\theta]}{2}\frac{f(x_n)}{f'(x_n)} + \frac{\theta(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)} - \frac{(1-\theta)m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}, \tag{14}$$

For the family of methods defined by (14), we have the following analysis of convergence.

Theorem 2.1. *Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable function $f : I \to \mathbf{R}$ for an open interval I. If x_0 is sufficiently close to α, then the methods defined by (14) are cubically convergent for any real value of θ, and satisfies the error equation*

$$e_{n+1} = K_3 e_n^3 + O(e_n^4), \tag{15}$$