Computers and Mathematics with Applications 68 (2014) 486–507

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Primal stabilized hybrid and DG finite element methods for the linear elasticity problem

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ARTICLE INFO

Article history:
Received 30 August 2013
Received in revised form 5 June 2014
Accepted 16 June 2014
Available online 2 July 2014

Keywords:
Linear elasticity
Discontinuous Galerkin
Hybridization
Stabilization
Local post-processing

ABSTRACT

Primal stabilized hybrid finite element methods for the linear elasticity problem are proposed consisting of locally discontinuous Galerkin problems in the primal variable coupled to a global problem in the multiplier which is identified with the trace of the displacement field. Numerical analysis, covering both continuous or discontinuous interpolations of the multiplier, shows that the proposed formulation preserves the main properties of the associate DG method such as consistency, stability, boundedness and optimal rates of convergence in the energy norm, and in the $\mathbf{L}^2(\Omega)$ norm for adjoint consistent formulations. Convergence studies confirm the optimal rates of convergence predicted by the numerical analysis presented here and a local post-processing technique is proposed to recover stress approximations with improved rates of convergence in $\mathbf{H}(\text{div})$ norm.

1. Introduction

Classical displacement based finite element methods for the elasticity problem determine the displacement field directly and evaluate the stresses by post-processing. The advantage of displacement-based formulation over a more complex mixed approach is that the introduction of additional unknowns and related difficulties are avoided. The disadvantages of displacement formulations are their well-known limitations, for example, the poor accuracy of the recovered stress approximations given by standard post-processing. For various reasons, mixed finite element methods in stress and displacement fields appear to be a natural choice. The pair forms a unique saddle point of the Hellinger–Reissner functional but, due to the symmetry constraint on the stress tensor, it is extremely difficult to construct stable finite element spaces which satisfy Brezzi's stability conditions [1,2]. In two spatial dimensions, the first stable finite element with polynomial shape functions is presented in Arnold and Winther [3,4] and extended in [5–9]. Stable mixed finite elements with weakly imposed symmetry [10–14], stabilized formulations [15–17] and nonconforming elements [18–20] have been also developed.

Discontinuous Galerkin (DG) methods are naturally a suitable alternative for numerically solving linear elasticity problems. Robustness, local conservation and flexibility for implementing h and p-adaptivity strategies are well known advantages of DG methods stemming from the use of finite element spaces consisting of discontinuous piecewise polynomials. A Local DG (LDG) method for linear elasticity is presented in [21] and interior penalty DG methods are considered in [22–25]. Mixed DG methods are given in [26,27] and a mixed DG method with weakly imposed symmetry of the stress tensors is

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http://dx.doi.org/10.1016/j.camwa.2014.06.014
0898-1221/

proposed in [28]. However, the practical utility of DG methods has been limited by their more complex formulation, computational implementation and much larger number of degrees-of-freedom they require compared with classical continuous Galerkin methods.

For elliptic problems, the natural connection between DG formulations and hybrid methods has been successfully exploited to derive new finite element methods with improved stability and reduced complexity and computational cost but still keeping the robustness and flexibility of DG methods [29–35]. A formulation that uses local, element-wise problems to project a continuous finite element space into a given discontinuous space, and then applies a discontinuous Galerkin formulation called Multiscale Discontinuous Galerkin (MDG) method introduced in [36] and analyzed in [37]. Arruda et al. [38] proposed the Locally Discontinuous but Globally Continuous (LDGC) finite element formulation combining the advantages of discontinuous Galerkin methods with the element based data structure and reduced computational cost of classical conforming finite element methods. Differently from the classical primal hybrid formulation of Raviart and Thomas [39], where the multiplier is identified with the flux, the LDGC multiplier is the trace of the primal variable. Similar works are the hybrid mortar method proposed and analyzed by Egger in [40] and the interface stabilized method developed by Labeur and Wells in [41] and analyzed by Wells in [42] for advection–diffusion–reaction equations.

Following Arruda et al. [38] and Egger in [40], we propose here a primal Stabilized Hybrid DG (SHDG) method for the linear elasticity problem in which the multiplier, identified with the trace of the displacement field, can be continuous as in [38] or discontinuous as in [40]. The method is proved to be stable for any order of interpolations of the displacement field and the multiplier. The local problems, in the displacement field, can always be solved at the element level in favor of the Lagrange multiplier and, consequently, the global system is assembled involving only the degrees of freedom associated with the multipliers. A numerical analysis, covering continuous or discontinuous interpolations of the multiplier, shows that the proposed formulation preserves the main properties of the associate DG method such as consistency, stability, boundedness and optimal rates of convergence in the energy norm, and in the $\mathbf{L}^2(\Omega)$ norm for adjoint consistent formulations. Stress approximations with observed optimal rates of convergence in $\mathbf{H}(\text{div})$ norm are obtained by a local post-processing of both displacement and stress using the multiplier approximation and residual forms of the constitutive and equilibrium equations at the element level.

The remainder of the paper is organized as follows. In Section 2 we present a review of notation and our model problem. The primal SHDG method is introduced in Section 3. A numerical analysis of the SHDG formulation is presented in Section 4 showing that it preserves the main properties of the associated DG method. In Section 5 the local and global problems are analyzed considering two strategies for solving the coupled system of linear equations associated with the SHDG formulation and a local post-processing is introduced to recover more accurate stress approximations. Numerical results on convergence studies are presented in Section 6. Concluding remarks are drawn in Section 7.

2. Preliminaries

2.1. Notation

To introduce the stabilized hybrid formulation, we first present some definitions and notations. Let $\Omega \in \mathbb{R}^d,\ d \geq 1$, be a bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$, and $L^2(\Omega)$ the space of square integrable functions, equipped with the usual norm $\|\cdot\|_{0,\Omega}$. Let $H^m(\Omega)$ be the usual Sobolev space of all functions in $L^2(\Omega)$ whose weak derivatives up to the nonnegative integer order m are also $L^2(\Omega)$-integrable [43]. The corresponding $H^m(\Omega)$ norms and semi-norms are denoted by $\|\cdot\|_{m,\Omega} = \|\cdot\|_m$ and $|\cdot|_{m,\Omega} = |\cdot|_m$, respectively. We also use the Hilbert space

$$\mathbf{H}(\text{div}) = \{\mathbf{u} \in [L^2(\Omega)]^d;\ \text{div}\,\mathbf{u} \in L^2(\Omega)\}$$

with norm

$$\|\mathbf{u}\|^2_{\mathbf{H}(\text{div})} := \|\mathbf{u}\|_0^2 + \|\text{div}\,\mathbf{u}\|_0^2.$$

For a given function space $V(\Omega)$, let $\mathbf{V}(\Omega) = [V(\Omega)]^d$ and $\mathbb{V}(\Omega) = [V(\Omega)]^{d\times d}$ be the spaces of all vector and tensor fields whose components belong to $V(\Omega)$. These spaces are furnished with the usual product norms (which, for simplicity, are denoted similarly as the norm in $V(\Omega)$). For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, and matrices $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}$, we use the standard notation. Furthermore, let $\mathbf{v}\otimes\mathbf{w}$ be the tensor that satisfy the following identity

$$\mathbf{v}\cdot\boldsymbol{\sigma}\mathbf{w} = \boldsymbol{\sigma} : (\mathbf{v}\otimes\mathbf{w}).$$

For simplicity we restrict our finite element formulation to two-dimensional domain Ω. Let

$$\mathcal{T}_h = \{K\} := \text{union of all elements } K$$

be a regular finite element partition of Ω and let

$$\mathcal{E}_h = \{e : e \text{ is an edge of } K \text{ for all } K \in \mathcal{T}_h\}$$

denote the set of all edges of all elements K of the mesh $\mathcal{T}_h$.

$$\mathcal{E}_h^0 = \{e \in \mathcal{E}_h\ e \text{ is an interior edge}\}$$