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Error analysis of a fully discrete finite element variational multiscale method for the natural convection problem[☆]

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ABSTRACT

For the natural convection problem, we propose a new projection-based finite element variational multiscale method by defining the stabilization terms via two local Gauss integrations at the element level. Based on the implicit backward Euler and implicit Crank–Nicolson schemes for temporal discretization and stabilized mixed finite element spatial discretization, we establish two numerical schemes for the natural convection problem. Unconditional stabilities of the two numerical schemes are proved. We derive error bounds of the fully discrete solution which are first and second order in time, respectively. The optimal error estimates in space could be achieved for velocity and temperature in the H^1 semi-norm, and for pressure in the L^2 norm with the proper choosing of stabilized parameters. However, the error estimates in space are suboptimal for velocity and temperature in the L^2 norm. The derived theoretical results are supported by two numerical examples.

1. Introduction

Natural convection is present in many real situations, such as room ventilation, double glass window design, etc. More importantly, it is behind the oceanic and atmospheric dynamics. Typically, fluid flow and heat transfer are governed by the partial differential equation system of mass, momentum and energy conservation, but in the case of natural convection the so-called Boussinesq approximation is generally employed. In this article, a finite element variational multiscale (VMS)

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method based on two local Gauss integrations is applied to solve numerically the time-dependent buoyancy driven flows, namely, the natural convection problem. The natural convection problem which we consider is: for bounded, polyhedral domains $\Omega_e \subset \Omega$ in $\mathbb{R}^d$ $(d = 2, 3)$ with $\mathrm{dist}(\partial\Omega_e, \partial\Omega) > 0$, the simulation time t^*, and the force field $\gamma : \Omega \times (0, t^*] \to \mathbb{R}$, find the velocity $u : \Omega \times (0, t^*] \to \mathbb{R}^d$, the pressure $p : \Omega \times (0, t^*] \to \mathbb{R}$ and the temperature $T : \Omega \times (0, t^*] \to \mathbb{R}$ satisfying [1,2]

$$u_t - Pr\,\Delta u + (u \cdot \nabla)u + \nabla p = Pr\,Ra\zeta T, \quad \zeta = g/|g|, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega_e, \qquad u \equiv 0 \quad \text{in } \Omega - \Omega_e = \Omega_s, \tag{2}$$

$$u|_{t=0} = u_0, \qquad \nabla \cdot u(x, t) = 0 \quad \text{in } \Omega_e, \tag{3}$$

$$T_t - \nabla \cdot (k\nabla T) + (u \cdot \nabla)T = \gamma \quad \text{in } \Omega, \tag{4}$$

$$T = 0 \quad \text{on } \Gamma_T, \qquad \frac{\partial T}{\partial n} = 0 \quad \text{on } \Gamma_B, \tag{5}$$

$$T|_{t=0} = T_0, \quad \text{in } \Omega, \tag{6}$$

where ζ is a unit vector in the direction of gravitational acceleration, γ is a known forcing function, n is the outward unit normal to Ω, and $\Gamma_T = \partial\Omega \setminus \Gamma_B$ where Γ_B is a regular open subset of $\partial\Omega$. Pr, Ra and $k > 0$ denote Prandtl number, Rayleigh number and the thermal conductivity parameter, respectively. Moreover, $k = k_e$ in Ω_e and $k = k_s$ in Ω_s, where k_e and k_s are positive constants. A global-in-time existence result for a more general natural convection problem (Navier–Stokes/Fourier model) can be found in Theorem 3.1 of [3].

The natural convection problem (or conduction–convection problems) includes not only the incompressibility and strong nonlinearity, but also the coupling between the energy equation and the equations governing the fluid motion. There are numerous works devoted to the development of efficient schemes for the natural convection problem ([1,4–21] and the references therein). We mention only a few papers here. Early papers on the stationary case are [6,7] by using the mixed finite element method. Cibik and Kaya [8] have formulated a projection-based stabilization finite element technique for solving the steady-state natural convection problems. The global stabilizations are added for both velocity and temperature variables and these effects are subtracted from the large scales. Keith J. Galvin et al. [9] have considered the problem of poor mass conservation in mixed finite element algorithms for flow problems with large rotation-free forcing in the momentum equation. Zhang et al. [10] have presented a subgrid stabilized defect-correction method for steady-state natural convection problem. Shi and Ren [11,12] have proposed a new stable nonconforming mixed finite element and a least squares Galerkin–Petrov nonconforming mixed finite element method for stationary conduction–convection problems. Boland and Layton [1] have derived stability properties and error estimates for the Galerkin finite element spatial discretization case when used to approximate heat flow in a fluid enclosed by a solid medium. Luo et al. [18] have given an optimizing reduced Petrov–Galerkin least squares mixed finite element for the non-stationary conduction–convection problem. Manzari [19] has used a standard Galerkin finite element method for spatial discretization and an explicit multistage Runge–Kutta scheme to march in the time domain for convection heat transfer problems. Benítez and Bermúdez [21] have presented a second order Lagrange–Galerkin method for natural convection problems.

If problem (1)–(6) with large Rayleigh number is solved by the usual Galerkin finite element method, it may exhibit global spurious oscillations [4,5] and yield inaccurate approximation. One reason is the dominance of the convection term. There are many numerical methods devoted to solving such problem, for example, the two-level stabilization scheme in [22], the defect-correction methods in [23–27,10], and the variational multiscale (VMS) method [28–35]. The original motivation of VMS methods was to justify the so-called stabilized finite element methods, which define the large scales in a different way, namely by projection into appropriate subspaces. The two local Gauss integrations method was first developed to offset the discrete pressure space by the residual of the simple and symmetry term at the element level in order to circumvent the inf–sup condition (see e.g., [36–38]), while the idea of two local Gauss integrations has been considered to deal with the VMS methods [39,16]. A significant feature of the two Gauss VMS method [39,16] is that the stabilization terms are defined by the difference between a consistent and an under-integrated matrix only involving the velocity gradient (and temperature gradient), rather than the projection operator as the common VMS methods used. The two local VMS method need not introduce extra variables and can keep the same accurate as the common VMS method. Shang [40] has presented an error analysis of a fully discrete finite element VMS method based on two local Gauss integrations for time-dependent Navier–Stokes equations. In this report, we will extend the new VMS method to a more complicated model: time-dependent natural convection problem. We have derived the stability and the error estimates of two fully discrete schemes, and given mathematical guidance on the selection of the methods' algorithmic parameters.

The paper is organized as follows. In Section 2, we introduce notation and mathematical preliminaries necessary for the analysis that follows. Two numerical schemes and their stabilities are presented in Section 3. In Section 4, error estimates of velocity, temperature and pressure are given for backward Euler temporal discretization. In Section 5, error estimates of velocity, temperature and pressure are given for Crank–Nicolson temporal discretization. Numerical experiments are presented in Section 6, including a convergence rate verification, and a test on natural convection cavity with left hand side heating that shows the two methods are effective at capturing large-scale behavior. Conclusions follow.