Computers and Mathematics with Applications 68 (2014) 269–287

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An efficient computational method for solving fractional biharmonic equation

M.H. Heydari *, M.R. Hooshmandasl, F.M. Maalek Ghaini

Faculty of Mathematics, Yazd University, Yazd, Iran
The Laboratory of Quantum Information Processing, Yazd University, Yazd, Iran

ARTICLE INFO

Article history:
Received 27 February 2014
Received in revised form 26 May 2014
Accepted 1 June 2014
Available online 20 June 2014

Keywords:
Shifted Chebyshev polynomials
Biharmonic equation
Operational matrix of fractional derivative

ABSTRACT

In this paper, we first introduce the fractional biharmonic equation and an orthogonal system of basis functions for the space of continuous functions on the interval $[0, L]$, generated by the shifted Chebyshev polynomials. Moreover, we propose a computational method based on the operational matrix of fractional derivatives of these basis functions for solving the fractional biharmonic equation. The main characteristic behind this approach is that it reduces the problem under consideration to solving a system of algebraic equations which greatly simplifies the problem. Convergence of the shifted Chebyshev polynomials expansion in two-dimensions is investigated. Also the power of this manageable method is illustrated.

1. Introduction

An important class of partial differential equations (PDEs) which arises in both physics and engineering is the biharmonic equation, i.e.:

$$\begin{cases} \triangle^2 u = f, & \text{on } \Omega, \\ u = g, & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial n} = h, & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, Ω is the bounded simply connected domain of which Γ is its boundary, $\frac{\partial u}{\partial n}$ is the outward normal derivative on Γ, and f, g and h are the known functions.

The biharmonic equation arises in the modeling of many engineering applications. For example, the bending behavior of a thin elastic rectangular plate, as might be encountered in ship design and manufacture, or the equilibrium of an elastic rectangle can be formulated in terms of the biharmonic equation. Also Stokes flow of a viscous fluid in a rectangular cavity under the influence of the motion of the walls can be described in terms of the solution of this equation. A more recent application of the biharmonic equation has been in the area of geometric and functional design, where it has been used as a mapping to produce efficient mathematical descriptions of surfaces in a physical space. It is worth mentioning that problems involving high-order PDEs are more difficult to solve than the second-order PDEs, so numerical methods are of significant

* Corresponding author at: Faculty of Mathematics, Yazd University, Yazd, Iran.
E-mail addresses: heydari@stu.yazduni.ac.ir, heydari@stu.yazd.ac.ir (M.H. Heydari), hooshmandasl@yazd.ac.ir (M.R. Hooshmandasl), maalek@yazd.ac.ir (F.M. Maalek Ghaini).

http://dx.doi.org/10.1016/j.camwa.2014.06.001
0898-1221/

interest and importance in the study of numerical solutions for them. In recent years, several approaches for the numerical solution of the biharmonic equation have been considered [1–12].

Fractional differential equations are generalized from integer order ones, which are obtained by replacing integer order derivatives by fractional order ones [13]. In comparison with integer order differential equations, the fractional differential equations show many advantages over the simulation of natural physical processes and dynamical systems [14–19]. The utility of fractional partial differential equations in mathematical modeling has attracted much attention in recent years [20]. There are different effective methods for solving fractional partial differential equations such as the fractional complex transform, the homotopy perturbation method, the variational iteration method, the heat-balance integral method and others (see [20] and references therein). Recently Chen et al. have proposed the Kansa method which belongs to the RBF collocation method for solving fractional diffusion equations [21]. In [22] Guang et al. proposed the finite difference method for solving variable-order time fractional diffusion equation. In [23] Fu et al. have proposed the Laplace transformed boundary particle method for solving time fractional diffusion equations. In this paper we consider the biharmonic equation (1) with fractional derivatives in the domain Ω, concerning the solution $u(x, y)$ satisfying the equation:

$$\begin{cases} \triangle^2_{\alpha,\beta}\, u = f, & \text{on } \Omega, \\ u = g, & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial n} = h, & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\triangle_{\alpha,\beta} = {}_xD^{\alpha}_{*} + {}_yD^{\beta}_{*}$, and the fractional order is $1 < \alpha,\ \beta \le 2$, and

$${}_xD^{\alpha}_{*}u(x, y) = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u''(s, y)}{(x-s)^{\alpha-1}} ds, \tag{3}$$

and

$${}_yD^{\beta}_{*}u(x, y) = \frac{1}{\Gamma(2-\beta)} \int_0^y \frac{u''(x, s)}{(y-s)^{\beta-1}} ds, \tag{4}$$

are fractional Caputo derivatives [24] with respect to the space variables x and y, respectively. It is mentionable that Caputo's derivative has the useful property as, $D^{\alpha}_{*}c = 0$, in which c is a constant and also

$${}_xD^{\alpha}_{*}x^n = \begin{cases} 0, & n < \lceil\alpha\rceil,\ n \in \mathbb{Z}^+ \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \ge \lceil\alpha\rceil,\ n \in \mathbb{Z}^+, \end{cases} \tag{5}$$

where we use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. For more details about fractional calculus and its properties see [24].

A usual way to solve functional equations is to assume that the unknown solution of the problem can be approximated by a linear combination of the basis functions. These basis functions can be for instants orthogonal or non orthogonal. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for the problems under study. Therefore, approximation by orthogonal families of basis functions has been found to be of wide applications in science and engineering. In recent years, the most commonly used orthogonal families of functions are sine–cosine functions, block pulse functions, Legendre, Chebyshev and Laguerre polynomials and also orthogonal wavelets for example Haar, Legendre, Chebyshev and CAS wavelets [25–34]. The main advantages of using an orthogonal basis is that the problem under consideration is reduced to a system of linear or nonlinear algebraic equations. This act not only simplifies the problem enormously but also speeds up the computational work during the implementation. This work can be done by truncating the series expansion in orthogonal basis functions for the unknown solution of the problem and using the operational matrices [34]. There are two main approaches for numerical solution of fractional differential equations:

One approach is based on converting the underlying fractional differential equations into fractional integral equations, and using the operational matrix of fractional integration, to eliminate the integral operations and reducing the problem into solving a system of algebraic equations. Another useful approach is based on using the operational matrix of fractional derivative to reduce the problem under consideration into a system of algebraic equations, and solving this system to obtain the numerical solution of the problem. The operational matrix of fractional derivative is given by:

$${}_xD^{\vartheta}_{*}\Psi(x) \simeq D^{\vartheta}\Psi(x), \tag{6}$$

where $\Psi(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_M(x)]^T$, in which $\psi_i(x)$ $(i = 0, 1, \ldots, M)$ are orthogonal basis functions which are orthogonal with respect to a specific weight function on a certain interval $[a, b]$ and D^{ϑ} is the operational matrix of fractional derivative of order ϑ of $\Psi(x)$. It is well known that we can expand any smooth function in terms of the eigenfunctions of certain singular Sturm–Liouville problems such as Legendre, Chebyshev, Laguerre and Hermite orthogonal polynomials. In this manner, the truncation error approaches zero faster than any negative power of the number of basic functions which is used in the expansion [35]. This phenomenon is usually referred to as "exponential rates of convergence/spectral accuracy" [35].