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First order system least squares method for the interface problem of the Stokes equations

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ABSTRACT

The first order system least squares method for the Stokes equation with discontinuous viscosity and singular force along the interface is proposed and analyzed. First, interface conditions are derived. By introducing a physical meaningful variable such as the velocity gradient, the Stokes equation transformed into a first order system of equations. Then the continuous and discrete norm least squares functionals using Legendre and Chebyshev weights for the first order system are defined. We showed that continuous and discrete homogeneous least squares functionals are equivalent to appropriate product norms. The spectral convergence of the proposed method is given. A numerical example is provided to support the method and its analysis.

1. Introduction

In many applications in fluid and bio-fluid mechanics, it is necessary to solve the incompressible Stokes or Navier–Stokes equations with discontinuous viscosity and singular forces. One example is Peskin's immersed boundary model that was introduced to simulate the blood flow in a human's heart [1]. In this paper, we consider the Stokes equation with discontinuous viscosity and singular force, that can be written as

$$\begin{cases} -\nu\Delta \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{g}\delta_\Gamma & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is the velocity vector, p is the pressure, and $\mathbf{f}$ is the external force function. The domain Ω is an open bounded domain separated into two sub-domains Ω_1 and Ω_2, by curve Γ, such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \Gamma$. Here, Γ is referred to as *interface*. The boundary of Ω is denoted by $\partial\Omega$. Let $\partial\Omega_1 = \overline{\Omega_1} \cap \partial\Omega$ and $\partial\Omega_2 = \overline{\Omega_2} \cap \partial\Omega$. The function $\mathbf{g}$ is force density defined only on the interface Γ and δ_Γ is 2-dimensional delta function with the support along the interface Γ. We assume that the viscosity ν is piecewise constant, defined by

$$\nu(x,y) = \begin{cases} \nu_1, & \text{if } (x,y) \in \Omega_1, \\ \nu_2, & \text{if } (x,y) \in \Omega_2. \end{cases}$$

The uniqueness of p can be achieved by imposing average zero, i.e., $\int_\Omega p dx = 0$. The existence and uniqueness of the weak solution of (1.1) can be found in [2,3]. It is well known that pressure is discontinuous and velocity continuous but non-smooth along the interface, due to the presence of singular source term and discontinuous viscosity coefficient. We call it as the *Stokes interface problem*. A lot of methods such as finite difference, finite element and finite volume method have been

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proposed for the case of continuous viscosity with singular source term without least squares method [4]. However, for the discontinuous viscosity, the jump condition for velocity and pressure is coupled, and getting accurate numerical solution is quite difficult. To overcome this problem, a plethora of works have been done to get accurate approximate solution [5,4]. Peskin's immersed boundary model that was introduced to study the fluid dynamics of blood flow in the human heart [1] is one of the most successful Cartesian grid methods. The method has been developed and applied to many biological problems. The immersed boundary method [6–8] has been used for problems with non-smooth but continuous solution which is only first order accurate. The immersed interface method which is second order accurate was developed by Leveque and Li [9] for the elliptic interface problem and then generalized to the Stokes problem [4]. The authors in [10] introduced two augmented variables that are defined only along the interface so that the jump conditions can be decoupled and the immersed interface method can be applied [9]. They got the second order immersed interface method using finite difference discretization. Rutka [11] developed the explicit jump immersed interface method (EJIIM) for two-dimensional Stokes flows on irregular domains which is up to second order derivatives along the interface. The authors in [12], using finite volume methods, reshaped immersed boundary cells and used polynomial interpolating functions to approximate the fluxes and gradients on the faces of the boundary cells which is second order accurate. However, in order to get second order accuracy in the above-mentioned methods, the jump condition in solution and the normal derivative are needed. Interface conditions for continuous viscosity can be found in [13,4] and for discontinuous viscosity in [5]. These interface conditions include the coupled interface condition for pressure and velocity, and zero's, first and second order derivative of velocity and pressure. The numerous number of interface conditions as well as being coupled make it difficult to use the first order system least squares method for the Stokes interface problem. In this paper, we derive interface conditions which include: zero's order for velocity (continuity of velocity) and the first order coupled interface condition for pressure and velocity. We use only two interface conditions and get accurate results. Y. Cao and M. D. Gunzburger [14] used the least squares finite element method to approximate the solution of elliptic interface problem by adding integral of interface conditions to the least squares functional. Authors in [15] used the first order system least squares method for Stokes–Darcy flow with Beavers–Joseph–Saffman conditions, by adding L^2-norm residual of interface conditions to the least squares functional. In a recent work in [16], authors derived optimal convergence of higher order finite element methods for elliptic interface problems in which error estimates are expressed in terms of the approximation order and a parameter δ that quantifies the mismatch between the smooth interface and the finite element mesh.

In one hand, the accuracy of spectral methods, which employ the global polynomial for discretization, makes it a popular method to approximate solutions of partial differential equations. The spectral collocation method also has been used to approximate the solution of interface problems. Shin and Jung [17] presented the spectral collocation method for one-dimensional interface problems. Hessari and Shin [18] have developed an algorithm to approximate the solutions of second order elliptic interface problems. On the other hand, the least squares method has received much attention in past decades, due to its advantageous features. Among the advantages of least squares method is that the choice of approximation spaces for velocity and pressure is not subject to the LBB compatibility condition and one can use the equal order interpolation polynomials to approximate all variables. In addition, the algebraic system which must be solved to compute the discrete solution is always symmetric and positive definite and can be easily preconditioned. This allows us to use an efficient iterative scheme such as the preconditioned conjugate gradient method.

I combined least squares and spectral collocation methods to approximate the solution of the incompressible Stokes equations (1.1). To employ least squares spectral collocation method for the Stokes interface problem, I extend the methodology presented in [19,20]. To do this, I first apply finite element argument to the problem (1.1), to derive interface conditions and Stokes equations in each sub-domain separately. The Stokes equation in each sub-domain is transformed into the first order system and then extended by some identities. The least squares functional is defined by summing up the squared L^2-norm of residual of extended first order system and squared L^2-norm of coupled jump condition for pressure and velocity, scaled by the viscosity constant. The jump condition for velocity (continuity of velocity across interface) is imposed into the velocity solution space. Actually, the velocity solution space includes the essential boundary condition and velocity jump condition. The homogeneous least squares functionals are shown to be equivalent to appropriate norm and the methods have spectral convergence. We also note that the analysis given here is for arbitrary domain Ω; however, the numerical experiment is done only for rectangle domain with straight line interface. In the case of curved interface, one can use the Gordon–Hall transformation (see [21,22,18,23] for more details).

The content of this paper is organized as follows. In the following section, we give some preliminaries which are useful in the sequel. Interface conditions are derived in Section 3. The first order system of equation for the Stokes interface problem is given in Section 4, including the ellipticity and coercivity of the least squares functional. Section 5 includes the discrete first order system least squares method and its spectral error discretization. Implementation and numerical experiment are presented in Section 6. We finalize the paper in Section 7 by some conclusions.

2. Preliminaries

In this section, we use the standard notations and definitions for the weighted Sobolev spaces $H^s_w(\mathcal{D})$, $s \geq 0$ associated to weighted inner products $(\cdot,\cdot)_{s,w}$, and respective weighted norms $\|\cdot\|_{s,w}$, where $w(x,y) = \hat{w}(x)\hat{w}(y)$ is either the Legendre weight function with $\hat{w}(t) = 1$ or the Chebyshev weight function with $\hat{w}(t) = \frac{1}{(1-t^2)^{1/2}}$, where $\mathcal{D} = (-1,1)^2$. The space