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Hamilton cycles in circuit graphs of matroids☆

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Abstract

In the circuit graph of a matroid the vertices are the circuits and the edges are the pairs CC' such that C and C' have nonempty intersection. It is proved that the circuit graph of a connected matroid with at least four circuits is uniformly Hamilton.

Keywords: Matroid; Circuit graph of matroid; Hamilton cycle; Uniformly Hamilton; Critical matroid

1. Introduction

We assume familiarity with graph theory and matroid theory. Terms and notation not defined here can be found in [1] for graphs and in [2] for matroids. A collection $\mathscr{C}$ of subsets of a finite set E is the set of circuits of a matroid M on E if and only if the following conditions (to be called circuit axioms) are satisfied:

(C1) If C_1, C_2 are distinct circuits, then $C_1 \nsubseteq C_2$.

(C2) If C_1, C_2 are circuits and $z \in C_1 \cap C_2$ there exists a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) - z$.

Let e be any element of E. Then we use M/e and $M \setminus e$ to denote the matroid obtained from M by contracting and deleting e, respectively. The family of circuits of $M \setminus e$ is those circuits of M which are contained in $E - e$. And the family of circuits of M/e is the family of sets that are minimal nonempty intersections of $E - e$ with circuits of M.

Matroid theory dates from the 1930's and Whitney in his basic paper [3] conceived a matroid as an abstract generalization of a matrix. Matroid theory gives us powerful techniques for understanding combinatorial optimization problems and for designing polynomial-time algorithms. In order to study the properties of circuits of matroids, we give a concept as follows. The *circuit graph* of a matroid $M = (E, I)$ is a graph $G = G(M)$ with vertex set $V(G)$ and edge set $E(G)$ such that $V(G) = \mathscr{C}$ and $E(G) = \{CC' \mid C, C' \in \mathscr{C}, |C \cap C'| \neq 0\}$, where the same notation is used for the vertices of G and the circuits of M. It is easy to see that the circuit graph of a matroid M without any coloop is connected if and only if M is connected.

A graph is *Hamilton* if it contains a Hamilton cycle. We now call a graph G *positively Hamilton*, written as $G \in H^+$, if every edge of G is in some Hamilton cycle; G is *negatively Hamilton*, written as $G \in H^-$, if for each

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edge of G there is a Hamilton cycle avoiding it. When $G \in H^+$ and $G \in H^-$, we say that G is *uniformly Hamilton*. Some other notation can be found in [1,2].

Maurer discussed the relationship of bases of matroids and graphs and defined the base graph of a matroid [4,5]. Alspach and Liu studied the properties of paths and cycles in base graphs of matroids [6]. Liu considered the connectivities of base graphs of matroids [7,8]. Harary considered the properties of tree graphs [9,10]. Recently Li, Bian and Liu studied the properties of matroid base incidence graphs [11]. Other related results can be found in [12–14]. In this paper we study the properties of cycles in the circuit graphs of matroids.

2. Preliminary results

We now state five lemmas which are used in the proofs in Section 3. Lemmas 1–4 can be found in [2], and Lemma 5 is due to Murty [15].

Lemma 1 (*[2]*). *A matroid M is connected if and only if for every pair e_1, e_2 of distinct elements of E, there is a circuit containing both e_1 and e_2.*

Lemma 2 (*[2]*). *If M is a connected matroid, then for every $e \in E$, either M/e or $M \setminus e$ is also connected.*

Lemma 3 (*[2]*). *Let C and $C^\star$ be any circuit and cocircuit of a matroid M. Then $|C \cap C^\star| \neq 1$.*

Lemma 4 (*[2]*). *If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$ where $C_1, C_2 \in \mathscr{C}$, then there exists a $C_3 \in \mathscr{C}$ such that $b \in C_3 \subseteq (C_1 \cup C_2) - \{a\}$.*

Let $M = (E, I)$ be a connected matroid. An element e of E is called an essential element if $M \setminus e$ is disconnected. Otherwise it is called a nonessential element. A connected matroid each of whose elements is essential is called a critically connected matroid or simply a critical matroid.

Lemma 5 (*[15]*). *A critical matroid of rank $\geqslant 2$ contains a cocircuit of cardinality 2.*

3. Main results

A matroid M is trivial if it has no circuits. In the following all matroids will be nontrivial.

Next we will discuss the properties of the matroid circuit graph. To prove the main results we firstly present the following remark which is clearly true.

Remark 1. Let M be any nontrivial matroid on E and $e \in E$. If G and G_1 are circuit graphs of M and $M \setminus e$, respectively, then G_1 is a subgraph of G induced by V_1 where $V_1 = \{C \mid C \in \mathscr{C}, e \notin C\}$. And the subgraph G_2 of G induced by $V_2 = V - V_1 = \{C \mid C \in \mathscr{C}, e \in C\}$ is a complete graph. $V(G_1)$ and $V(G_2)$ partition $V(G)$.

Lemma 6. *For any matroid $M = (E, I)$ which has a cocircuit $\{a, b\}$, then $G(M) \cong G(M/a)$.*

Proof. Since $|C \cap \{a, b\}| \neq 1$ for any circuit C, by Lemma 3, the circuits of M can be partitioned into two classes, those circuits containing both a and b and those circuits containing neither a nor b. Likewise, the circuits of M/a can be partitioned into two classes: those containing b and those not containing b; clearly there is a bijection between $\mathscr{C}(M)$ and $\mathscr{C}(M/a)$. Hence $G(M) \cong G(M/a)$. □

Lemma 7. *Suppose that $M = (E, I)$ is a connected matroid with an element e such that the matroid $M \setminus e$ is connected and $G = G(M)$ is the circuit graph of matroid M. Let $G_1 = G(M \setminus e)$ be the circuit graph of $M \setminus e$ and G_2 be the subgraph of G induced by V_2 where $V_2 = \{C \mid C \in \mathscr{C}, e \in C\}$. If the matroid $M \setminus e$ has more than one circuit, then for any edge $C_1C_2 \in E(G)$, there exists a 4-cycle $C_1C_2C_3C_4$ in graph G such that one edge of the 4-cycle belongs to $E(G_1)$ and one belongs to $E(G_2)$, and C_1, C_2 are both adjacent to C_3.*

Proof. By Remark 1, $V(G_1)$ and $V(G_2)$ partition $V(G)$. There are three cases to consider.

Case 1. If $e \in E - (C_1 \cup C_2)$, then C_1C_2 is an edge of $G(M \setminus e)$. By Lemma 4, there are at least three vertices in $G(M \setminus e)$. There is an element e_1 such that $e_1 \in C_1 \cap C_2$. Let G_1 and G_2 be the graphs defined as above. Note that