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Nonlinear triple-point problems with change of sign

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Abstract

In this paper, we study the existence of at least one or two positive solutions to the second-order triple-point nonlinear boundary value problem

$$y''(x) + h(x)f(y(x)) = 0, \quad x \in [a, b]$$
$$y(a) = \alpha y(\eta), \qquad y(b) = \beta y(\eta),$$

where $0 < \alpha < \beta < 1$ and $\eta \in (a, b)$. Here h changes sign in η. As an application, we also give some examples to demonstrate our results.

Keywords: Positive solutions; Fixed-point theorems; Cone; Alternating coefficient

1. Introduction

Three-point boundary value problems for differential equations have been studied in recent years. In most of these studies, the function h is assumed to be nonnegative or nonpositive (see [1–5]). Liu [6] has studied the existence of positive solutions of the second-order boundary value problem

$$\begin{cases} y''(x) + \lambda a(x)f(y(x)) = 0, \quad 0 < x < 1, \\ y(0) = 0, \qquad y(1) = \beta y(\eta), \end{cases} \tag{1.1$'$}$$

where λ is a positive parameter, $0 < \beta < 1, 0 < \eta < 1$, the function a is an alternating coefficient on $[0, 1]$. He used the Krasnoselskii fixed-point theorem and obtained some simple criteria for the existence of at least one positive solution of the BVP (1.1′).

In this paper, we shall use Krasnoselskii fixed-point theorem and Avery–Henderson fixed-point theorem to investigate the existence of at least one positive solution and of at least two positive solutions respectively to triple-point boundary value problem

$$\begin{cases} y''(x) + h(x)f(y(x)) = 0, \quad x \in [a, b], \\ y(a) = \alpha y(\eta), \qquad y(b) = \beta y(\eta), \end{cases} \tag{1.1}$$

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where $0 < \alpha < \beta < 1$, $a < \eta < b$, h changes sign in η.

We will assume that the following conditions are satisfied.

(H1) $f : [0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing.

(H2) $h : [a, b] \to \mathbb{R}$ is continuous and such that $h(x) \geq 0$, $x \in [a, \eta]$; $h(x) \leq 0$, $x \in [\eta, b]$. Moreover, it does not vanish identically on any subinterval of $[a, b]$.

(H3) There exists a constant $\tau \in (a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), \eta)$ such that, for all $x \in [0, b - \eta]$ the function

$$H(x) = \delta h^{+}(\eta - \delta x) - \frac{1}{\Lambda} h^{-}(\eta + x) \geq 0,$$

where $h^{+}(x) = \max\{h(x), 0\}$, $h^{-}(x) = -\min\{h(x), 0\}$, and

$$\delta = \frac{\eta - \tau}{b - \eta}, \qquad \Lambda = \frac{\beta - \alpha}{b - a} \min\left\{\frac{\beta}{1 - \alpha}(\eta - a), b - \eta, \frac{\alpha}{1 - \alpha}(b - a)\right\}.$$

Our (H3) condition is a generalization of the condition (H4) of Liu [6].

2. Preliminary lemmas

In this section, we present auxiliary lemmas which will be used later.

First, define the number D by

$$D = \alpha(\eta - b) + \beta(a - \eta) + b - a.$$

Lemma 2.1. *Let $D \neq 0$. Then for $k \in \mathcal{C}[a, b]$, the problem*

$$\begin{cases} y''(x) + k(x) = 0, & x \in [a, b], \\ y(a) = \alpha y(\eta), & y(b) = \beta y(\eta) \end{cases} \tag{2.1}$$

has the unique solution

$$y(x) = -\int_a^x (x - s)k(s)\mathrm{d}s + \frac{\alpha(x - b) + \beta(a - x)}{D} \int_a^{\eta} (\eta - s)k(s)\mathrm{d}s + \frac{\alpha(\eta - x) + x - a}{D} \int_a^b (b - s)k(s)\mathrm{d}s.$$

Let $G(x, s)$ be the Green's function for the problem (2.1). A direct calculation gives the following:

$$G(x, s) = \begin{cases} G_1(x, s), & a \leq x \leq \eta, \\ G_2(x, s), & \eta < x \leq b, \end{cases}$$

where

$$G_1(x, s) = \begin{cases} g_{11}(x, s) = \dfrac{[\beta(x - \eta) + b - x](s - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & a \leq s \leq x, \\ g_{12}(x, s) = \dfrac{\alpha(b - \eta)(s - x) + [\beta(s - \eta) + b - s](x - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & x < s \leq \eta, \\ g_{13}(x, s) = \dfrac{[\alpha(\eta - x) + x - a](b - s)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & \eta < s \leq b, \end{cases}$$

and

$$G_2(x, s) = \begin{cases} g_{21}(x, s) = \dfrac{[\beta(x - \eta) + b - x](s - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & a \leq s \leq \eta, \\ g_{22}(x, s) = \dfrac{(b - x)[\alpha(\eta - s) + s - a] + \beta(x - s)(\eta - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & \eta < s \leq x, \\ g_{23}(x, s) = \dfrac{[\alpha(\eta - x) + x - a](b - s)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & x < s \leq b. \end{cases}$$