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Weighted analytic regularity in polyhedra

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ABSTRACT

We explain a simple strategy to establish analytic regularity for solutions of second order linear elliptic boundary value problems. The abstract framework presented here helps to understand the proof of analytic regularity in polyhedral domains given in the authors' paper in [M. Costabel, M. Dauge, S. Nicaise, Analytic regularity for linear elliptic systems in polygons and polyhedra, Math. Models Methods Appl. Sci. 22 (8) (2012)]. We illustrate this strategy by considering problems set in smooth domains, in corner domains and in polyhedra.

1. Introduction

Solutions of elliptic boundary value problems with analytic data are analytic. This classical result has played an important role in the analysis of harmonic functions since Cauchy's time and in the analysis of more general elliptic problems since Hilbert formulated it as his 19th problem. Hilbert's problem for second order nonlinear problems in variational form in two variables was solved by Bernstein in 1904 [1]. After this, many techniques were developed for proving analyticity, culminating in the 1957 paper [2] by Morrey and Nirenberg on linear problems, where Agmon's elliptic regularity estimates in nested open sets were refined to get Cauchy-type analytic estimates, both in the interior of a domain and near analytic parts of its boundary.

Analyticity means exponentially fast approximation by polynomials, and therefore it plays an important role in numerical analysis, too. Analytic estimates have gained a renewed interest through the development of the p and h–p versions of the finite element method by Babuška and others [3,4]. In this context, applications often involve boundaries that are not globally analytic, but only piecewise analytic due to the presence of corners and edges, and therefore global elliptic regularity results cannot be used directly.

The way of proving such analytic regularity results (which we call here "type A" for short) is quite technical and often difficult to follow, as can be seen for example from papers by Babuška and Guo [5–7] devoted to corner domains. In order to tackle polyhedral domains with success, it was necessary to alleviate some difficulties as much as possible. In our paper [8] on type A results for polyhedral domains, we eventually completed a proof that had been missing for a long time. To do this, we relied on already known basic regularity results of low order (called here "type B") and proved what remains after that. In this paper, we present this new approach in a more abstract and systematic way, which shows how to attain this aim with as little effort as possible. The program consists in dividing the proof into two fundamental steps.

The first step involves results of type B, namely basic regularity results that exist for many boundary value problems in domains with different regularity properties. Such results are often well known, in some cases since a long time (for example [9] in 1959 for smooth domains and [10] in 1967 for corner domains), in others more recently (polyhedral domains [11,12]).

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The second step consists in proving "regularity shift" results, which we call "type S". In our context, they involve the proof of Cauchy type estimates for the derivatives of the solution at any order.

The abstract framework behind this approach can be summarized as follows:

$$\boxed{\text{Type B} + \text{Type S} \implies \text{Type A}}$$

We illustrate this strategy in the context of linear elliptic boundary value problems. We first recall some results for the case of domains with analytic boundary that are well known but can be proved with the help of our program. Second we present some analytic regularity results for corner domains, extending results obtained by Babuška and others for polygonal domains [5–7,13]. Finally we state recent results for polyhedral domains that we proved in [8], using anisotropic weighted Sobolev spaces introduced in [14,15].

2. An abstract framework

We study the question of regularity of the solutions of elliptic boundary value problems. More precisely, consider a boundary value problem, written in compact form with a linear operator $\mathbb{P}$ as

$$\mathbb{P}\boldsymbol{u} = \boldsymbol{q}$$

where $\boldsymbol{q}$ may include interior, boundary, or interface data.

A regularity statement takes the form

$$\boxed{\boldsymbol{u} \in \mathbb{U}_{\mathsf{base}} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Q}_{\mathsf{data}} \implies \boldsymbol{u} \in \mathbb{U}_{\mathsf{sol}}}$$

The ideal situation can be summarized as follows:

- $\mathbb{U}_{\mathsf{base}}$ is a space where existence of solutions is known,
- $\mathbb{U}_{\mathsf{sol}}$ is optimal in the sense that $\mathbb{P}$ is bounded from $\mathbb{U}_{\mathsf{sol}}$ into $\mathbb{Q}_{\mathsf{data}}$,
- if $\mathbb{Q}_{\mathsf{data}}$ is a space of piecewise analytic data, $\mathbb{U}_{\mathsf{sol}}$ is a space of piecewise analytic solutions.

In the literature, three types of relevant theorems can be found:

Type C (*Existence of solutions in a basic space* $\mathbb{V}$). *This is typically the consequence of a coercive variational formulation or, more generally, of a Fredholm alternative.*

Type B (*Basic regularity*)

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{B}}_{\mathsf{data}} \implies \boldsymbol{u} \in \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}}$$

for suitable $\mathbb{Q}^{\mathsf{B}}_{\mathsf{data}}$ and where $\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ is a space involving estimates on a finite number of derivatives (e.g. a space of strong solutions).

Type A (*Analytic regularity*)

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{A}}_{\mathsf{data}} \implies \boldsymbol{u} \in \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}}$$

for suitable $\mathbb{Q}^{\mathsf{A}}_{\mathsf{data}}$ and where $\mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ involves estimates on all derivatives with Cauchy-type growth.

As mentioned in the introduction, a fourth type of statement (Type S) plays a fundamental role in our strategy:

Type S (*Regularity shift*)

$$\boxed{\boldsymbol{u} \in \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{A}}_{\mathsf{data}} \implies \boldsymbol{u} \in \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}}$$

Our main strategy and idea is to use the scheme

$$\boxed{\text{Type B} + \text{Type S} \implies \text{Type A}}$$

Hence our remaining task is to find suitable pairs $(\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}, \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}})$ so that

1. A result of type B is known,
2. We are able to prove corresponding results of type S.

The spaces $\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ and $\mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ will be built with the help of a countable set of semi-norms

$$|\cdot|_{\mathbb{X}^\ell}, \quad \ell \in \mathbb{N}_0.$$

Typically, the semi-norm $|\cdot|_{\mathbb{X}^\ell}$ is a norm on derivatives ∂^α of order $|\alpha| = \ell$. With this sequence a full family of spaces can be associated in a natural way:

1. Finite regularity spaces for any natural number k

$$\mathbb{X}^k = \left\{\boldsymbol{u} : |\boldsymbol{u}|_{\mathbb{X}^\ell} < \infty, \forall \ell = 0, \dots, k\right\}$$

   associated with the norm $\|\boldsymbol{u}\|_{\mathbb{X}^k} = \max_{\ell=0,\dots,k} |\boldsymbol{u}|_{\mathbb{X}^\ell}$.