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An a-posteriori error estimate for *hp*-adaptive DG methods for convection–diffusion problems on anisotropically refined meshes

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ABSTRACT

We prove an a-posteriori error estimate for *hp*-adaptive discontinuous Galerkin methods for the numerical solution of convection–diffusion equations on anisotropically refined rectangular elements. The estimate yields global upper and lower bounds of the errors measured in terms of a natural norm associated with diffusion and a semi-norm associated with convection. The anisotropy of the underlying meshes is incorporated in the upper bound through an alignment measure. We present a series of numerical experiments to test the feasibility of this approach within a fully automated *hp*-adaptive refinement algorithm.

1. Introduction

We derive and numerically test a residual-based a-posteriori error estimate for *hp*-version discontinuous Galerkin (DG) methods for the convection–diffusion model problem:

$$\begin{aligned} -\varepsilon\Delta u + \underline{a}(x)\cdot\nabla u &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

Here, Ω is a bounded Lipschitz polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$. The parameter $\varepsilon > 0$ is the (constant) diffusion coefficient, the function $\underline{a}(x) \in W^{1,\infty}(\Omega)^2$ a given flow field, and $f(x)$ a source term in $L^2(\Omega)$. We assume that

$$\nabla\cdot\underline{a} = 0 \quad \text{in } \Omega. \tag{2}$$

For simplicity, we shall also assume that $\|\underline{a}\|_{L^\infty(\Omega)}$ and the length scale of Ω are of order one so that ε^{-1} is the Péclet number of the problem. The standard weak form of the convection–diffusion equation (1) is to find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = \int_\Omega \left(\varepsilon\nabla u\cdot\nabla v + \underline{a}\cdot\nabla u v\right) dx = \int_\Omega f v\, dx \quad \forall\, v \in H_0^1(\Omega). \tag{3}$$

Under assumption (2), the variational problem (3) is uniquely solvable.

This paper is a continuation of our work on *hp*-adaptive DG methods for diffusion and convection–diffusion problems. This work was initiated in [1], where an energy norm a-posteriori error estimate was derived for *hp*-version DG methods for diffusion problems in two dimensions. The key technical tool was the introduction of an *hp*-version averaging operator, inspired by that of [2] for *h*-version DG methods. In [3], related averaging techniques were used in the numerical analysis of continuous interior penalty *hp*-elements. Extensions to linear elasticity in mixed form, quasi-linear elliptic problems and three-dimensional diffusion equations were presented in [4–6], respectively. In [7], the same averaging approach was

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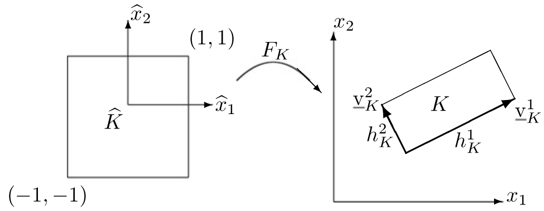

Fig. 1. Anisotropic directions of rectangle K.

pursued to derive an error estimator for *hp*-adaptive DG methods for convection–diffusion equations on isotropically refined meshes. This estimator has the distinct feature that it is robust in the Péclet number of the problem with respect to a suitably defined error measure (i.e., it is reliable and efficient with constants that are independent of the parameter ε).

The purpose of this paper is to extend the work [7] to anisotropically refined meshes, and to present an estimator η which yields global upper and lower bounds of the error measured in terms of a natural norm associated with diffusion and a semi-norm associated with convection. In particular, our error measure contains the standard DG energy norm and a variant of the dual norm introduced in [8] to measure convective effects. The constant in the lower bound is independent of ε and the mesh size, but weakly depending on the polynomial degrees, as in many *hp*-version error estimators for diffusion problems. In the upper bound, we use an alignment measure to incorporate the anisotropy of the underlying meshes in the reliability constant; see [9–11] and the references therein. As a consequence, the upper bound depends on the elemental aspect ratios and is not fully robust in the Péclet number, in contrast to the case of isotropic elements considered in [7]. Our analysis is valid for 1-irregularly refined rectangular elements with arbitrarily large aspect ratios, and is based on the *hp*-version averaging operator of [7], but with anisotropically scaled approximation properties.

We present a series of numerical experiments to test the feasibility of this approach within a fully automated *hp*-adaptive algorithm. Our tests indicate that internal and boundary layers are correctly captured and resolved at exponential rates of convergence in the number of degrees of freedom. We further observe that as soon as a reasonable *h*-resolution of the layers is achieved, the alignment measure is of moderate size, and the ratios of the error estimators and the energy errors are practically independent of the diffusion parameter ε and the mesh size. In all the tests, our new *hp*-version anisotropic refinement strategy outperforms similar strategies based on isotropic mesh refinement by orders of magnitude.

Let us also point out that in [12,13], a duality-based a-posteriori approach was successfully proposed and studied for *hp*-adaptive DG methods for convection–diffusion problems on anisotropically refined meshes and with anisotropically enriched elemental polynomial degrees.

The outline of the rest of the paper is as follows. In Section 2, we introduce *hp*-adaptive discontinuous Galerkin methods for the discretization of the convection–diffusion problem (1). In Section 3, we state and discuss our a-posteriori error estimates. The proof of these estimates is carried out in Section 4. In Section 5, we present a series of numerical tests illustrating the performance of a fully automated *hp*-adaptive algorithm. Finally, in Section 6, we end with some concluding remarks.

Throughout the paper, we shall frequently use the symbols $\lesssim$ and $\gtrsim$ to denote bounds that are valid up to positive constants, independently of the local mesh sizes, the elemental aspect ratios, the elemental polynomial degrees, and the parameter ε.

2. Interior penalty discretization

In this section, we introduce an *hp*-version interior penalty DG finite element method for the discretization of Eq. (1) on anisotropically refined meshes.

2.1. Elements and meshes

We consider (a family of) partitions $\mathcal{T}$ of Ω into disjoint rectangular elements $\{K\}$. Each element is the image of the reference square $\widehat{K} = (-1, 1)^2$ under an affine elemental mapping F_K. We allow for 1-irregularly refined meshes, where each elemental edge may contain at most one hanging node located in the middle of the edge. For each rectangle $K \in \mathcal{T}$, we denote by $\underline{v}_K^1$ and $\underline{v}_K^2$ its two anisotropic directions, as shown in Fig. 1. With the direction vectors, we associate the matrix

$$\mathbf{M}_K = [\underline{v}_K^1, \underline{v}_K^2]. \tag{4}$$

The lengths of the direction vectors are denoted by h_K^1 and h_K^2, respectively. Then we define the minimum and maximum diameters of an element K by

$$h_{\min,K} = \min\{h_K^1, h_K^2\}, \qquad h_{\max,K} = \max\{h_K^1, h_K^2\}. \tag{5}$$

We denote by $\mathcal{N}(K)$ the set of four vertices of K, and define $\mathcal{N}(\mathcal{T}) = \cup_{K\in\mathcal{T}} \mathcal{N}(K)$. We further split the set of all nodes into interior nodes and boundary nodes, that is, we write $\mathcal{N}(\mathcal{T}) = \mathcal{N}_I(\mathcal{T}) \cup \mathcal{N}_B(\mathcal{T})$. We denote by $\mathcal{E}(K)$ the set of four elemental edges of an element K. The length of an elemental edge is denoted by h_E, i.e., $h_E = h_K^i$ if E is parallel to $\underline{v}_K^i$, $i = 1, 2$.