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International Journal of Sediment Research 28 (2013) 77-89

INTERNATIONAL JOURNAL OF SEDIMENT RESEARCH
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A semi-implicit three-dimensional numerical model for non-hydrostatic pressure free-surface flows on an unstructured, sigma grid

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Abstract

A semi-implicit 3-D numerical formulation for solving non-hydrostatic pressure free-surface flows on an unstructured, sigma grid is proposed. Pressure-splitting and θ semi-implicit methods are inherited and reformed from Casulli's z-coordinate formulation. The non-orthogonal sigma-coordinate transformation leads to additional terms. The resulting linear system for the non-hydrostatic correction is diagonally dominant but unsymmetric, and it is solved by the BiCGstab method. In contrast with z-coordinate non-hydrostatic models, the new model fits vertical boundaries much better, which is important for the long-time simulation of sediment transport and riverbed deformation. A lock-exchange density flow is computed to determine whether the new scheme is able to simulate non-hydrostatic free-surface flows. The new model is further verified using the field data of a natural river bend of the lower Yangtze River. Good agreement between simulations and earlier research results, field data is obtained, indicating that the new model is applicable to hydraulic projects in real rivers.

Key Words: Three-dimensional, Non-hydrostatic pressure, Semi-implicit, Sigma-coordinate, Unstructured grid

1 Introduction

The hydrostatic pressure distribution assumption is usually employed to reduce the complexity and computational burden of full pressure–velocity coupling problems in earlier three-dimensional (3-D) hydrodynamic models for free-surface flows (Blumberg and Mellor, 1987). However, this assumption is no longer valid for flows over abruptly-changing bed topographies, flows with sharp density gradients or short-wave motions, where the ratio of the vertical scale to horizontal scale of motion is not sufficiently small (Kocyigit et al., 2002). Moreover, although hydrostatic models can respond to the presence of bed forms and give a total resistance similar to that given by non-hydrostatic models, they do not capture the flow separation at the crest of dunes (Kheiashy et al., 2010).

When including non-hydrostatic pressure effects in hydrodynamic models, it is essential to involve a costly solution of the 3-D Poisson equation. There were few such attempts before 1995 owing to the limited computer power at that time. Since then, 3-D non-hydrostatic models for free-surface flows have been intensively studied and developed in line with the rapid improvement in computer capacity. Many non-hydrostatic models (Casulli and Stelling, 1995; Marshall et al., 1997; Casulli and Stelling, 1998; Casulli and Zanolli, 2002; Yuan and Wu, 2004; Hu et al., 2009) adopt the z-coordinate. A linear system for the discrete Poisson equation with a positively-defined and symmetric coefficient matrix is easily solved by sophisticated iterative solvers. However, it is difficult for the z-coordinate to fit the vertical boundaries well. The fixed and step-like division of the vertical domain may depress the computation accuracy and introduce stability problems in modeling sediment transport and riverbed deformation.

To better fit the vertical boundaries, researchers adopt the sigma coordinate, which maps the irregular physical domain between the free surface and uneven bottom to a regular computational domain (Mahadevan et al., 1996; Jankowski, 1999; Li and Fleming, 2001; Lin and Li, 2002; Kocyigit et al., 2002; Kanarska and Maderich, 2003; Heggelund et al., 2004; Bradford, 2005; Lee et al., 2006; Berntsen and Xing, 2006; Zhang and Liu, 2006; Young and Wu, 2007; Hu et al., 2011). At the same time, the non-orthogonal sigma-coordinate transformation introduces additional terms. The discretization and calculation of these terms adds extra complexity, error and computational burden to the numerical model. These issues have been addressed in various ways, leading to several typical formulations (a short review of

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Note: The original manuscript of this paper was received in Sept. 2011. The revised version was received in Dec. 2011. Discussion open until Mar. 2014.

sigma-coordinate non-hydrostatic models is given in Appendix A). However, it is still an open issue to pursue an accurate, stable and efficient numerical formulation for modeling non-hydrostatic pressure free-surface flows including timely, variable vertical boundaries.

We extend Casulli's (2002) z-coordinate non-hydrostatic model to its sigma-coordinate version in this paper. The sediment transport model and its applications will be reported in later papers. The remaining sections are organized as follows. Governing equations and boundary conditions are introduced in section 2; the numerical formulation is described in section 3; verifications of the 3-D non-hydrostatic model are presented in section 4; discussions and conclusions are given in section 5.

2 Governing equations and boundary conditions

In the following, the z- and sigma-coordinate systems are respectively denoted by (x*, y*, z*, t*) and (x, y, σ, t). Applying the pressure splitting, the total pressure $p(x^*, y^*, z^*, t^*)$ is decomposed into three parts: the barotropic and baroclinic contributions to the hydrostatic pressure and the non-hydrostatic pressure. The pressure splitting is expressed as

$$p(x^*,y^*,z^*,t^*) = g\int_z^{H_R+\eta}\rho_0 dz + g\int_z^{H_R+\eta}(\rho-\rho_0)dz + \rho_0 q(x^*,y^*,z^*,t^*) \tag{1}$$

where ρ and ρ_0 are respectively the water density and reference density, kg m^{-3}; g is gravitational acceleration, m s^{-2}; H_R is the height of the undisturbed water surface, m; $h(x^*, y^*, t^*)$ and $\eta(x^*, y^*, t^*)$ are respectively the bathymetry and water elevation measured from the undisturbed water surface, m; p and q are respectively the total and non-hydrostatic pressure of the water, m^{-2} s^{-2}.

Denoting the water depth $D = h + \eta$, the transformation of the sigma-coordinate (x, y, σ, t) is given by $\sigma = (z^* - \eta)/D$. The transformation maps the irregular vertical physical domain between the free surface and the uneven bottom to the regular computational domain [–1, 0]. The 3-D Reynolds-averaged Navier–Stokes equations with Boussinesq approximation for incompressible fluids in the sigma-coordinate system are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial \sigma}=fv-g\frac{\partial \eta}{\partial x}-\frac{g}{\rho_0}\int_{z^*}^{H_R+\eta}\frac{\partial \rho}{\partial x^*}dz^*+\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{K_{mv}}{D}\frac{\partial u}{\partial \sigma}\right)+K_{mh}\left(\frac{\partial^2 u}{\partial x^{*2}}+\frac{\partial^2 u}{\partial y^{*2}}\right)-\left[\frac{\partial q}{\partial x}+\frac{\partial q}{\partial \sigma}\frac{\partial \sigma}{\partial x^*}\right] \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial \sigma}=-fu-g\frac{\partial \eta}{\partial y}-\frac{g}{\rho_0}\int_{z^*}^{H_R+\eta}\frac{\partial \rho}{\partial y^*}dz^*+\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{K_{mv}}{D}\frac{\partial v}{\partial \sigma}\right)+K_{mh}\left(\frac{\partial^2 v}{\partial x^{*2}}+\frac{\partial^2 v}{\partial y^{*2}}\right)-\left[\frac{\partial q}{\partial y}+\frac{\partial q}{\partial \sigma}\frac{\partial \sigma}{\partial y^*}\right] \tag{3}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\omega\frac{\partial w}{\partial \sigma}=\frac{1}{D}\frac{\partial}{\partial \sigma}\left(\frac{K_{mv}}{D}\frac{\partial w}{\partial \sigma}\right)+K_{mh}\left(\frac{\partial^2 w}{\partial x^{*2}}+\frac{\partial^2 w}{\partial y^{*2}}\right)-\frac{1}{D}\frac{\partial q}{\partial \sigma} \tag{4}$$

subject to the continuity equation

$$\frac{\partial \eta}{\partial t}+\frac{\partial uD}{\partial x}+\frac{\partial vD}{\partial y}+D\frac{\partial \omega}{\partial \sigma}=0 \tag{5}$$

where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ are respectively the velocity components in the horizontal x^*-direction and y^*-direction and vertical z^*-direction, m s^{-1}; t is time, s; f is the Coriolis parameter, s^{-1}; K_{mh} and K_{mv} are respectively the coefficients of the horizontal and vertical eddy viscosity, m^2 s^{-1}; and ω is the vertical velocity in the sigma-coordinate system and is given by

$$\omega=\frac{d\sigma}{dt^*}=\frac{w}{D}-u\left(\frac{\sigma}{D}\frac{\partial D}{\partial x}+\frac{1}{D}\frac{\partial \eta}{\partial x}\right)-v\left(\frac{\sigma}{D}\frac{\partial D}{\partial y}+\frac{1}{D}\frac{\partial \eta}{\partial y}\right)-\left(\frac{\sigma}{D}\frac{\partial D}{\partial t}+\frac{1}{D}\frac{\partial \eta}{\partial t}\right) \tag{6}$$

Integrating the continuity equation from the bottom defined at σ = –1 to the free-surface at σ = 0 and using the kinematic condition at the free-surface yields the depth-integrated continuity equation, namely the free surface equation (FSE):

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[D\int_{-1}^{0}ud\sigma\right]+\frac{\partial}{\partial y}\left[D\int_{-1}^{0}vd\sigma\right]=0 \tag{7}$$

Equations (2)–(5) and (7) constitute a set of equations for the velocity components u, v and w, the free surface η and the non-hydrostatic pressure q. The transformation of the horizontal unstructured grid does not change the form of the equations, and the above governing equations are also applicable in the local coordinate system of the horizontal unstructured grid.

A state equation of the form $\rho = \rho(C)$ is used to relate the water density to the concentration "C"of a conservative scalar, and the system is then closed. At the riverbed, the balance between the internal Reynolds stress and the bottom friction shear stress gives the boundary condition

$$\rho_0\frac{K_{mv}}{D}\left(\frac{\partial u}{\partial \sigma},\frac{\partial v}{\partial \sigma}\right)_b=(\tau_{bx},\tau_{by}) \tag{8}$$

where the bottom friction stress is defined as $(\tau_{bx},\tau_{by})=\rho_0 C_D\sqrt{u_b^2+v_b^2}(u_b,v_b)$; u_b and v_b are the velocity components of the lowest grid, m s^{-1}; and C_D is the bottom friction coefficient (Zhou et al., 2009).