Computers and Mathematics with Applications 71 (2016) 2513–2523

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions of a linear matrix equation

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ARTICLE INFO

Article history:
Received 11 September 2015
Received in revised form 17 December 2015
Accepted 26 March 2016
Available online 4 May 2016

Keywords:
Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions
Least squares solution
Matrix equation

ABSTRACT

In this paper, we first establish the necessary and sufficient conditions for the existence and the explicit expressions of the Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions of the matrix equation $AX = B$, and meanwhile the best approximation solution is considered. Then, if the solvability conditions are not satisfied, the least squares Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions and the least squares Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions with the minimum norm of the above matrix equation are respectively derived. In addition, two algorithms are shown to compute the least squares Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions, and the corresponding numerical examples are also given to illustrate the feasibility of the algorithms.

1. Introduction

Throughout the paper, the following notations and symbols will be used. The set of all $m \times n$ complex matrices, the set of all $n \times n$ Hermitian matrices and the set of all $n \times n$ unitary matrices are respectively denoted by $\mathbb{C}^{m\times n}$, $\mathbb{C}^n_H$ and $U\mathbb{C}^{n\times n}$. The symbol A^* represents the conjugate transpose of the matrix $A \in \mathbb{C}^{m\times n}$. I_n stands for the identity matrix of order n. $(A \quad B)$, $\begin{pmatrix} A \\ B \end{pmatrix}$ and $A \circ B$ denote a row block matrix, a column block matrix and the Hadamard product produced by A and B, respectively. For two matrices $A, B \in \mathbb{C}^{m\times n}$, the inner product is defined by $\langle A, B\rangle = tr(B^*A)$. The norm $\|\cdot\|$, induced by the inner product, is called the Frobenius norm. The Moore–Penrose generalized inverse $A^\dagger$ of matrix $A \in \mathbb{C}^{m\times n}$ is defined to be the unique solution $X \in \mathbb{C}^{m\times n}$ of the system of matrix equations

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA.$$

Moreover, R_A and L_A mean the two orthogonal projectors $R_A = I_m - AA^\dagger$ and $L_A = I_n - A^\dagger A$, which satisfy $R_{A^*} = L_A$ and $L_{A^*} = R_A$.

The reflexive and anti-reflexive matrices have been widely applied into system and control theory, engineering, scientific computations and other fields (see, e.g., [1–4]), and the reflexive and anti-reflexive solutions of a linear matrix equation or systems of matrix equations have been also studied by many authors (see, e.g., [5–13]). For instance, Peng and Hu [11] gave the necessary and sufficient conditions for the existence of and the explicit expressions for the reflexive and anti-reflexive solutions of the classical matrix equation

$$AX = B. \tag{1.1}$$

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http://dx.doi.org/10.1016/j.camwa.2016.03.021
0898-1221/

As the extension of the (anti-)reflexive matrix, the generalized (anti-)reflexive solutions of the matrix equation(s) were established (see, e.g., [14,15]). Then, the (P,Q) generalized (anti-)reflexive solutions,the $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions and the $\{P, Q, \mathrm{k}+1\}$-(anti-)reflexive solutions of the matrix equation(s) were deeply studied, see, e.g., [16–18]. As the special cases of the $\{P, \mathrm{k}+1\}$-reflexive and $\{P, \mathrm{k}+1\}$-anti-reflexive matrices, the following two types of matrices are defined.

Definition 1.1. Let $P \in \mathbb{C}^{n\times n}$ be a Hermitian $\{\mathrm{k}+1\}$-potent matrix, that is, $P^* = P = P^{k+1}$. Then $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian $\{P, \mathrm{k}+1\}$-reflexive matrix if $A = A^* = PAP$.

Definition 1.2. Let $P \in \mathbb{C}^{n\times n}$ be a Hermitian $\{\mathrm{k}+1\}$-potent matrix, that is, $P^* = P = P^{k+1}$. Then $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian $\{P, \mathrm{k}+1\}$-anti-reflexive matrix if $A = A^* = -PAP$.

The sets of all $n \times n$ Hermitian $\{P, \mathrm{k}+1\}$-reflexive matrices and all $n \times n$ Hermitian $\{P, \mathrm{k}+1\}$-anti-reflexive matrices are respectively denoted by $HR\mathbb{C}^{n\times n}$ and $HAR\mathbb{C}^{n\times n}$. Note that the matrix $P \in \mathbb{C}^{n\times n}$, which appears in the following, is always the Hermitian $\{\mathrm{k}+1\}$-potent matrix.

The well-known linear matrix equation (1.1) has been widely and deeply investigated by many authors (see, e.g., [18–32]). For example, Groß [19] obtained the explicit solution of Eq. (1.1). Khatri and Mitra [20] constructed the Hermitian and nonnegative definite solution of Eq. (1.1). Kyrchei [21,22] studied the least squares solutions and the minimum norm least squares solutions of Eq. (1.1). Wang and Yu [28] derived the generalized bi(skew-)symmetric solutions of Eq. (1.1). Zhao, Hu and Zhang [32] got its bisymmetric least squares solutions under a central principal submatrix constraints.

By the work mentioned above, especially the work in [16,17], we in this paper are motivated to solve the following three problems. Notice that there has been little information about them until now.

Problem 1. Construct the solvable conditions and the explicit expressions of the Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions of Eq. (1.1).

Problem 2. Given $\widehat{X} \in \mathbb{C}^{n\times n}$. Find $\overline{X} \in K$ such that

$$\left\|\widehat{X} - \overline{X}\right\| = \min_{X\in K} \left\|\widehat{X} - X\right\|,$$

where K is the solution set of Problem 1.

Problem 3. Find the least squares Hermitian $\{P, \mathrm{k}+1\}$-(anti-)reflexive solutions of Eq. (1.1), and provide the algorithms to compute the least squares solutions. Then, some examples are presented to show the efficiency of the algorithms.

The paper is organized as follows. In Section 2, some well-known results are presented. In Section 3, the necessary and sufficient conditions for the existence and the explicit expressions of the Hermitian $\{P, \mathrm{k}+1\}$-reflexive solutions and the Hermitian $\{P, \mathrm{k}+1\}$-anti-reflexive solutions of Problem 1 are respectively explored. In Section 4, the solutions of Problem 2 are investigated. In Section 5, Problem 3 is solved. In Section 6, some conclusions are made.

2. Preliminaries

In this section, some important lemmas are introduced to give main help to solve Problems 1–3.

Lemma 2.1 ([15,16]). *Let $P \in \mathbb{C}_H^n$. Then P is a Hermitian $\{\mathrm{k}+1\}$-potent matrix if and only if P is idempotent when k is odd, or tripotent when k is even. Moreover, there exists $U \in U\mathbb{C}^{n\times n}$ such that*

$$P = U\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}U^*, \quad \text{if } k \text{ is odd (i.e., } P \text{ is idempotent)},$$

or

$$P = U\begin{pmatrix} I_r & 0 & 0 \\ 0 & -I_{s-r} & 0 \\ 0 & 0 & 0 \end{pmatrix}U^*, \quad \text{if } k \text{ is even (i.e., } P \text{ is tripotent)},$$

where $s = r(P)$.

Remark 2.1. From the above lemma, it can be seen that the Hermitian $\{\mathrm{k}+1\}$-potent matrix P can be reduced to only two forms: $P^2 = P(\mathrm{k}=1)$ and $P^3 = P(\mathrm{k}=2)$, which have been obtained in [16]. Hence, we in this paper just establish the Hermitian $\{P, 2\}$-reflexive solutions and the Hermitian $\{P, 3\}$-(anti-)reflexive solutions of Eq. (1.1).

Similar to the proof of the decompositions of the $\{P, 2\}$-reflexive matrix and the $\{P, 3\}$-reflexive matrix in [16], the following three results can be easily obtained.