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An expandable local and parallel two-grid finite element scheme

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ABSTRACT

An expandable local and parallel two-grid finite element scheme based on superposition principle for elliptic problems is proposed and analyzed in this paper by taking example of Poisson equation. Compared with the usual local and parallel finite element schemes, the scheme proposed in this paper can be easily implemented in a large parallel computer system that has a lot of CPUs. Convergence results based on H^1 and L^2 a priori error estimation of the scheme are obtained, which show that the scheme can reach the optimal convergence orders within $|\ln H|^2$ or $|\ln H|$ two-grid iterations if the coarse mesh size H and the fine mesh size h are properly configured in 2-D or 3-D case, respectively. Some numerical results are presented at the end of the paper to support our analysis.

1. Introduction

Two-grid or multi-grid finite element methods and domain decomposition methods are powerful tools for numerical simulation of solutions to PDEs with high resolution, which are otherwise inaccessible due to the limits in computational resources. For examples, the domain decomposition schemes, nonlinear Galerkin schemes and two-grid/two-level post-processing schemes in [1–10] and the references therein. In the past decade, a local and parallel two-grid finite element method for elliptic boundary value problems was initially proposed in [11] and was extended to nonlinear elliptic boundary value problems in [12] and Stokes and Navier–Stokes equations in [13,14].

Let us briefly recall the local and parallel two-grid finite element method in [11] for the following simple Poisson equation with Dirichlet boundary condition defined in convex domain $\Omega \subset R^d,\ d = 2, 3$:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

whose weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{1.2}$$

Suppose $T^H(\Omega)$ is a regular coarse mesh triangulation of Ω and $S^H(\Omega) \subset H^1(\Omega)$, $S_0^H(\Omega) = S^H(\Omega) \cap H_0^1(\Omega)$ are the corresponding finite element spaces defined on $T^H(\Omega)$. Let us decompose the entire domain Ω into a series of disjoint subdomains, $\overline{\Omega} = \bigcup_{j=1}^{N} \overline{D_j}$. For example, see Fig. 1.

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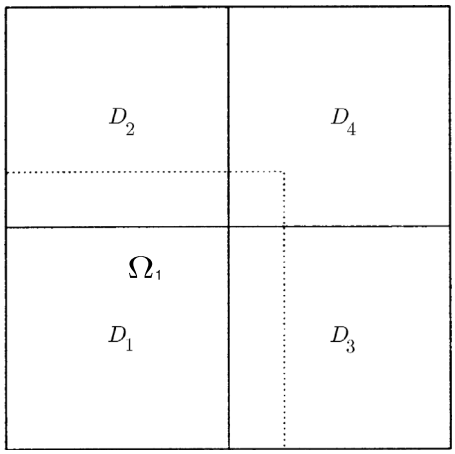

Fig. 1. Decomposition of the domain Ω.

If the coarse mesh standard Galerkin approximation $u_H \in S_0^H(\Omega)$ is obtained, by expanding each subdomain D_j to another subdomain $\Omega_j \subset \Omega$ and for a given fine mesh size $h < H$, one of the local and parallel two-grid schemes proposed in [11] is: find $e_h^j \in S_0^h(\Omega_j)$ such that

$$a(e_h^j, v) = (f, v) - a(u_H, v), \quad \forall v \in S_0^h(\Omega_j). \tag{1.3}$$

And the final approximation u^h is defined piecewisely by

$$u^h = u_H + e_h^j, \quad \text{in } D_j,\ j = 1, 2, \ldots, N.$$

Error estimations in [11] show that u^h can reach the optimal convergence order in H^1 norm. However, it is obvious that u^h is in general discontinuous and its L^2 error bound does not in general have higher order than its H^1 error bound. To overcome this defect of the algorithm, the authors in [11] modified the above scheme to ensure the continuity of u^h in Ω and finally do a coarse grid correction to get the optimal error bound in L^2 norm. The most attractive feature of the algorithm is that the series of subproblems are independent once u_H is known and therefore it is a highly parallelized algorithm.

On the other hand, one can easily see from the content of [11] that their error estimates heavily depend on the usage of the superapproximation property of finite element spaces. Thanks to [15], we know that the usage of this property makes the error constant appeared in [11] has the form $O(t^{-1})$, where $t = \text{dist}(\partial D_j \setminus \partial\Omega, \partial\Omega_j \setminus \partial\Omega)$. To guarantee the error orders obtained in [11], one should demand that $t = O(1)$. That means the distance between the boundaries of a specific subdomain D_j and its expansion Ω_j should be of constant order. Therefore Ω_j could not be arbitrary small even when $\text{diam}(D_j)$ tends to zero. This will lead to a vast waste of parallel computing resources.

In this paper, we follow the basic idea presented in [11] to construct another form of two-grid local and parallel scheme, in which the scale of each subproblem can be much smaller compared with that in [11]. In fact, we will deal with the case of $\text{diam}(D_j) = O(H)$ and $t = O(H)$. We call the scheme an expandable local and parallel two-grid scheme because the scale of each subproblem can be arbitrary small as H tends to zero and every two adjacent subproblems only have a small overlapping. Similarly, to get a better L^2 error bound, a coarse grid correction is done in each cycle of two-grid iteration.

Different from the previously mentioned local and parallel schemes, we use superposition principle to generate a series of local and independent subproblems and this will make the global approximation continuous in Ω. Such kind of technique has been successfully used in [16,17], in which adaptive variational multi-scale methods were constructed. In fact, the scheme in this paper is quite similar to the variational multi-scale schemes in the two references. The difference is the schemes presented in [16,17] are adaptive schemes based upon some a posterior error estimates and therefore some boundaries related problems have to be solved. Another contribution of this paper compared with [16,17] is that a priori error estimate of the scheme is obtained and for patches of given size, our analysis shows that a few iterations, say $O(|\ln H|^2)$ or $O(|\ln H|)$ in 2-D or 3-D respectively, will generate an approximation with same accuracy as the fine mesh standard Galerkin approximation. In addition, following the idea of partition of unity method (see [18]), authors in [19,20] proposed a local and parallel two-grid scheme for second order linear elliptic equations in 2-D case based on the scheme presented in [11]. Although the usage of partition of unity method makes the global approximation continuous, but their error estimation is still based on the superapproximation property of the finite element space and therefore the distance t, theoretically, must be constant order to guarantee their estimations.

The rest of this paper is organized as follows. In the coming section, some preliminary materials are provided. In Section 3, local and parallel scheme is constructed. Error estimates in both H^1 and L^2 norms are obtained for the scheme in Section 4. Finally, some numerical experiments are given to support our analysis in Section 5.