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Adaptive optimal control approximation for solving a fourth-order elliptic variational inequality[☆]

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ABSTRACT

An optimal control approach is proposed to solve the fourth-order elliptic variational inequality with curvature obstacle. It is proved that the variational inequality is equivalent to the constrained optimal control problem. The finite element approximation of the optimal control problem is constructed and the a priori error estimates and the equivalent a posteriori error estimators are derived. Some numerical experiments are performed to confirm a priori error estimates and demonstrate the effectiveness of the a posteriori estimators.

1. Introduction

Fourth-order elliptic variational inequalities with obstacles have been widely studied, such as variational inequalities with curvature obstacle, displacement obstacle, and so on. There have been many researches for solving these variational inequalities. For example, Noor and Al-Said used the finite difference method in [1,2], and they developed a new cubic spline method for computing the approximate solution of a system of fourth-order boundary value problems associated with an obstacle in [3]. Glowinski, Lions and Tremolieres used penalty and relaxation methods in [4,5], Momania, Moadia and Noor used the decomposition method in [6], Shi and Wang used non-conforming finite element methods in [7–11] to solve these fourth order variational inequalities with obstacles. Also, Brézis and Stampacchia studied the regularity of a fourth order variational inequality with curvature obstacle with two type of boundary conditions in [12]. Up to now, it is still a difficult and interesting problem to solve a fourth-order variational inequality effectively.

The purpose of this article is to develop a new approach to solve a fourth-order elliptic variational inequality with curvature obstacle. The idea is to use the adaptive optimal control approach to obtain the solution indirectly. To this end, we translate the fourth-order variational inequality with curvature obstacle into the equivalent form, which contains two Poisson equations and a 2rd order variational inequality. The optimality condition we obtained in this paper is equivalent to the mixed variational form deduced by Deng and Shen in [13]. In [13], they have the priori error estimates for $\|y - y_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ with the order $h^{1-\varepsilon},\ \forall \varepsilon > 0$, but in our paper, we obtained the h^1 order convergence for the above two error estimates.

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The paper is organized as follows: The notations used throughout the paper are introduced in Section 2, together with the equivalence proof between the fourth-order variational inequality and the constrained optimal control problem, and also the optimality conditions are given. In Section 3, we present the finite element approximation of the control problem. In Section 4, we give the priori error estimate of y. Equivalent a posteriori error estimates with H^1 norm and L^2 norm are deduced in Section 5. In Section 6, we give two numerical experiments to demonstrate our error estimates developed in Sections 4 and 5.

2. Fourth-order variational inequality and its equivalent form

Let Ω be a convex domain in R^2 with the Lipschitz boundary $\partial\Omega$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for the Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$. In addition, c or C denotes a general positive constant independent of h.

2.1. Fourth-order variational inequality

Let $K = \{y \in H^2(\Omega) \cap H_0^1(\Omega) : \Delta y \le 0 \text{ in } \Omega\}$. Here, the condition satisfied in K is called the curvature obstacle. We consider the following fourth-order variational inequality problem: *Find y in K such that*

$$\int_\Omega \Delta y \Delta(w - y) \ge \int_\Omega f(w - y), \quad \forall w \in K \tag{2.1}$$

where $f \in L^2(\Omega)$ is a given function.

In engineering, the variational inequality problem (2.1) describes the curvature obstacle problem. In general it is very difficult to solve the fourth-order variational inequality problem. In the next subsection, we translate the variational inequality problem (2.1) into an equivalent optimal control problem, which is governed by the second order PDE, so that it can be solved easily by using known-well adaptive finite element methods.

2.2. Equivalent optimal control problem

Define a convex set K' of the form:

$$K' = \{u \in L^2(\Omega);\ u \ge 0 \text{ (a.e.) in } \Omega\}.$$

It is clear that K' is closed in $L^2(\Omega)$. We formulate the optimal control problem equivalent to the fourth-order variational inequality problem (2.1) in the following equivalent theorem.

Theorem 2.1. *The problem (2.1) is equivalent to the following optimal control problem:*

$$\begin{cases} \min\limits_{u\in K'} \left\{ J(y, u) = \dfrac{1}{2}\displaystyle\int_\Omega u^2 - \int_\Omega fy \right\}, \\ s.t. \ -\Delta y = u \quad in\ \Omega,\ y = 0 \ on\ \partial\Omega. \end{cases} \tag{2.2}$$

Proof. It is clear that the minimized problem (2.2) may be represented by

$$\min_{y\in K} \left\{ J(y) = \frac{1}{2}\int_\Omega (\Delta y)^2 - \int_\Omega fy \right\}. \tag{2.3}$$

Obviously, J is convex. Suppose y is the solution of the minimized problem (2.3). From [5], we know the following conclusion:

$$J(y) = \min_{y\in K} J(y) \Longleftrightarrow J'(y)(w - y) \ge 0, \quad \forall w \in K.$$

Since

$$J'(y)(w - y) = \int_\Omega \Delta y \Delta(w - y) - \int_\Omega f(w - y) \ge 0, \quad \forall w \in K,$$

we deduce that $J'(y)(w - y) \ge 0$ for each $w \in K$ is equivalent to (2.1). Then Theorem 2.1 is proved. □

Theorem 2.1 shows that one could solve the optimal control problem (2.2) instead of solving the variational inequality problem (2.1). In the following parts we study how to solve the optimal control problem (2.2). To this end, we give a weak formula for the state equation. Let the state space $V = H_0^1(\Omega)$ and the control space $U = L^2(\Omega)$. Set

$$a(y, w) = \int_\Omega \nabla y \cdot \nabla w \quad \forall y, w \in V; \qquad (f, g) = \int_\Omega fg \quad \forall f,\ g \in U.$$