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Multi-soliton and double Wronskian solutions of a (2 + 1)-dimensional modified Heisenberg ferromagnetic system

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ABSTRACT

A (2 + 1)-dimensional modified Heisenberg ferromagnetic system, which arises in the motion of magnetization vector of the isotropic ferromagnet and biological pattern formation, is investigated. Via the Hirota bilinear method, multi-soliton solutions of such a system are derived. It is proved that the system possesses the N-soliton solutions expressed in terms of the double Wronskian determinant. Head-on and overtaking elastic interactions are exhibited. Elastic interaction behavior between the two solitons has been interpreted through the asymptotic analysis, namely, amplitude and velocity of each soliton remain unchanged except for the phase shift after the interaction. Inelastic interactions including the soliton fusion and fission between two solitons are shown. During the soliton propagation, for the product of two fields, the soliton with the smaller amplitude can travel faster than with the larger, while for the third field, the soliton with the larger amplitude can travel faster than with the smaller. On the other hand, the soliton for the third field may exhibit the solitoff-like property. With respect to the three solitons, head-on elastic interaction can be found.

1. Introduction

Heisenberg-type models for the spin–spin interactions have been proposed to explain the magnetic ordering in the ferromagnetic materials [1–10]. For instance, one-dimensional J_1–J_2 Heisenberg models (with the ferromagnetic nearest-neighbor coupling J_1 and antiferromagnetic next–nearest-neighbor coupling J_2) can be used to describe the properties for the family of the quasi-one-dimensional edge-shared chain cuprates such as $LiCu_2O_2$, Li_2ZrCuO_4 and Li_2CuO_2 [4,5], while the two-dimensional spin-$\frac{1}{2}$ J_1–J_2 quantum Heisenberg models have been introduced to interpret the interplay of frustration effects and quantum fluctuations [6,7]. Besides, the Hersenberg-type models can be applied to the investigation on the magnetic solitons in isotropic/anisotropic ferromagnets, and on the motion of a nonplanar vortex in the circular easy-plane magnet with a rotating inplane magnetic field [8–10].

More generally, models for the nonlinear phenomena in fluid dynamics, nonlinear optics, Bose–Einstein Condensates, biological molecules, chemical systems, etc., can be the nonlinear evolution equations (NLEEs) [11–15]. As one kind of the NLEEs, nonlinear Schrödinger (NLS)-type equations can be used to describe the nonlinear water waves in fluids, ion-acoustic waves in plasmas, nonlinear envelope pulses in the fibers, pressure pulses in the artery vessels and nonlinear Rossby waves in the atmosphere [16–18]. Thereinto, the gauge equivalent counterpart of the so-called NLS^- equation

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(i.e., $i\,q_\tau + q_{\xi\xi} - 2|q|^2\,q = 0$, where q is the complex amplitude or envelope of the wave packet, while ξ and τ are the spatial and temporal variables) through a moving space curve formalism is the $(1+1)$-dimensional Heisenberg ferromagnetic (HF) model [19–21],

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \tag{1}$$

which arises from the motion of magnetization vector of the isotropic ferromagnet, where $\vec{S} = (S_1, S_2, S_3)$ is the spin unit vector, and "$\times$" represents the cross product, while x and t are the spatial and temporal variables.

To describe certain nonlinear phenomena, higher dimensional NLEEs have been proposed [22–29]. Due to the dependence on the additional spatial variables in higher dimensional NLEEs, more solution structures might be seen, such as the solitoffs, dromions, lumps, breathers and multi-valued solitons [22]. By virtue of the prolongation structure, higher dimensional integrable generalizations for some $(1+1)$-dimensional NLEEs can be derived [21]. An extension of Eq. (1) is the $(2+1)$-dimensional modified HF system [26–29], as follows:

$$u_t + u_{xy} - u\,w = 0, \tag{2a}$$

$$v_t - v_{xy} - v\,w = 0, \tag{2b}$$

$$w_x + (u\,v)_y = 0, \tag{2c}$$

which is associated with the $(2+1)$-dimensional NLS$^-$ equation $\left[\text{i.e., } i\,q_\tau + q_{\xi\eta} - 2\,q \int d\xi \left(|q|^2\,q\right)_\eta = 0, \text{ with } \eta \text{ the spatial variable}\right]$, where u, v and w are the functions of x, t, as well as spatial variable y. System (2) can also be used to model the biological pattern formation in reaction–diffusion process [29,30]. In Ref. [27], System (2) has been investigated through the prolongation structure and Lax representation. In Refs. [28,29], integrable property of System (2) has been studied via the Painlevé analysis, and some localized coherent and periodic solutions have been given by means of the multi-linear variable separation approach.

On the other hand, for revealing the nonlinear mechanisms, it is helpful to search for the analytic solutions and investigate the underlying dynamics of relevant NLEEs [31–34]. So far, some analytic methods have been proposed for constructing the soliton solutions, such as the inverse scattering transformation, bilinear method, Wronskian technique and Darboux transformation [11–15,31–34].

Among those approaches, the bilinear method provides a way to construct the analytic soliton solutions of a NLEE [35–52]. Based on the bilinear forms, the soliton solutions in terms of the Nth-order exponential polynomial, Wronskian, Gramm and Casorati determinants for some $(1+1)$- and $(2+1)$-dimensional NLEEs can be constructed [38–42]. Quasi-periodic and decay mode solutions can also be obtained [43–45]. In some cases, the linear combinations of exponential traveling waves (implying the existence of linear subspaces of solutions) can be derived [46,47]. Besides, the Wronskian technique can be implemented to validate the multi-soliton solutions, since each column of a Wronskian is the derivative of the previous one, and higher derivatives lead to the sums of the Wronskians [48–50]. Additionally, the bilinear method can be generalized to a kind of bilinear equations possessing the linear subspaces of solutions [51,52].

For System (2), to our knowledge, soliton solutions in terms of the Nth-order exponential polynomial, soliton interaction and Double Wronskian Solutions have not been reported as yet. Hereby, in Section 2, with the aid of symbolic computation [53,54], multi-soliton solutions of System (2) will be derived by means of the Hirota bilinear method, and the double Wronskian solutions will be given. In Section 3, soliton propagation and interaction will be investigated, including the head-on and overtaking elastic interactions, soliton fusion and fission. Section 4 will be devoted to the conclusions.

2. Multi-soliton solutions of system (2)

Via the dependent variable transformation,

$$u(x, y, t) = \frac{G}{F}, \qquad v(x, y, t) = \frac{H}{F}, \qquad w(x, y, t) = 2\,(\log F)_{xy}\,, \tag{3}$$

System (2) can be transformed into the following bilinear forms:

$$\left(D_t + D_x D_y\right) G \cdot F = 0, \tag{4a}$$

$$\left(D_t - D_x D_y\right) H \cdot F = 0, \tag{4b}$$

$$D_x^2\, F \cdot F + G H = 0, \tag{4c}$$

where $G(x, y, t)$, $H(x, y, t)$ and $F(x, y, t)$ are the analytic functions, and $D_x^l D_y^m D_t^n$ is the Hirota bilinear derivative operator [35–37] defined by

$$D_x^l D_y^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, y, t)\, b(x', y', t')\bigg|_{x'=x,\, y'=y,\, t'=t},$$
$$l, m, n = 0, 1, 2, \ldots, \tag{5}$$

with a and b as the analytic functions, while x', y' and t' are all independent variables.