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The numerical solution of scattering by infinite rough interfaces based on the integral equation method

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ABSTRACT

In this paper, we describe a Nyström integration method for the integral operator T which is the normal derivative of the double-layer potential arising in problems of two-dimensional acoustic scattering by infinite rough interfaces. The hypersingular kernel and unbounded integral interval of T are the key difficulties. By using a mollifier, we separately deal with these two difficulties and propose its Nyström integration method. Furthermore, we establish convergence of the method. Finally, we apply the method to the scattering problem by infinite rough interfaces and carry out some numerical experiments to show the validity.

1. Introduction

We study the direct scattering problem for infinite rough interfaces which was motivated by several applications such as underwater acoustics and radar techniques. See Fig. 1, we consider a situation where an infinite rough interface $\Gamma := \{x \in \mathbb{R}^2 : x_2 = f(x_1), x_1 \in \mathbb{R}\}$ separates the two media Ω_1 and Ω_2 from each other, where Ω_1 and Ω_2 denote the regions above and below the interface Γ, respectively. Suppose a plane wave $u^i(x) = \exp(ikx\cdot d)$ of direction $d = (-\cos\phi_0, -\sin\phi_0)$ with the angle of incidence $\phi_0 \in (0, \pi)$, an entire solution to the Helmholtz equation in $\mathbb{R}^2$, is incident on the infinite interface Γ from the top region Ω_1. Then one has to determine the unknown scattered wave u_1^s in Ω_1 and the unknown transmitted wave u_2^s in Ω_2 such that the total wave $u := u^i + u_1^s$ in Ω_1, $u := u_2^s$ in Ω_2 satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

with different wave numbers $k = k_1$ in Ω_1 and $k = k_2$ in Ω_2.

To ensure uniqueness, some radiation conditions along with growth conditions in the x_2 direction have to be imposed on the scattered waves $u_j^s (j = 1, 2)$. Throughout this paper we adopt the *upward and downward propagating radiation condition* (*UPRC* and *DPRC*) [1,2] with the following definition.

Definition 1.1. Given a domain $G \subset \mathbb{R}^2$, and letting v be a solution to the Helmholtz equation $\Delta v + k^2 v = 0$ in G, v is said to satisfy the upward (downward) propagating radiation condition UPRC (DPRC) in G if, for some $h \in \mathbb{R}$ and $\phi \in L^\infty(\Gamma_h)$,

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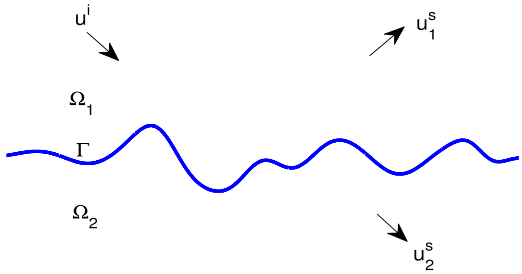

Fig. 1. Geometry of scattering by an infinite rough interface.

it holds that $U_h^+ \subset G(U_h^- \subset G)$ and

$$v(x) = 2\theta \int_{\Gamma_h} \frac{\partial \Phi_k(x,y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_h^+ \ (x \in U_h^-),$$

where $\theta = 1$ for the UPRC and $\theta = -1$ for the DPRC, and

$$\Phi_k(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x, y \in \mathbb{R}, \ x \neq y,$$

is the free-space Green's function for the Helmholtz operator $\Delta + k^2$, $\Gamma_h := \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 = h\}$, $U_h^\pm := \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 \gtrless h\}$.

We denote the set of functions satisfying the *UPRC* (*DPRC*) in G with the wave number k by $UPRC(G, k)$ ($DPRC(G, k)$). Next we present the growth condition in the x_2 direction: for some $\beta \in R$

$$\sup_{\Omega_j} |x_2|^\beta |u_j^s(x)| < \infty, \quad j = 1, 2. \tag{1.2}$$

Then the above transmission problem can be formulated as the following more general boundary value problem.

Boundary Value Problem (BVP): Given $g_1 \in BC^{1,\alpha}(\Gamma)$ and $g_2 \in BC^{0,\alpha}(\Gamma)$, find $u_1^s \in C^2(\Omega_1) \cap BC^1(\overline{\Omega_1} \setminus U_{h_1}^+)(h_1 > f_+)$ and $u_2^s \in C^2(\Omega_2) \cap BC^1(\overline{\Omega_2} \setminus U_{h_2}^-)(h_2 < f_-)$, satisfying (1.1), (1.2), $u_1^s \in UPRC(\Omega_1, k_1)$, $u_2^s \in DPRC(\Omega_2, k_2)$ and the transmission conditions

$$u_1^s - u_2^s = g_1, \qquad \partial_\nu u_1^s - \partial_\nu u_2^s = g_2, \quad \text{on } \Gamma. \tag{1.3}$$

Here, we abbreviate $\frac{\partial}{\partial \nu}$ to ∂_ν, where $\nu(x) = (\nu_1(x), \nu_2(x))$ stands for the unit normal vector at $x \in \Gamma$ pointing into Ω_1, $\tau(x) = (\nu_2(x), -\nu_1(x))$ denotes the tangent vector at the point $x \in \Gamma$, $f_+ := \sup_{x_1 \in \mathbb{R}} f(x_1)$ and $f_- := \inf_{x_1 \in \mathbb{R}} f(x_1)$ for a function f.

Before going further, we first introduce some useful notations. For $V \subset \mathbb{R}^2$, we denote by $BC(V)$ the set of functions bounded and continuous on V, a Banach space under the norm defined by $\|\psi\|_{\infty,V} := \sup_{x \in V} |\psi(x)|$. Note that $BC^n(\mathbb{R}^2)$ is the Banach space of all functions whose derivatives up to order n are bounded and continuous on $\mathbb{R}^2$. Given $v \in L^\infty(V)$, we denote by $\partial_j v$, $j = 1, 2$, the (distributional) derivative $\partial v(x)/\partial x_j$. For $0 < \alpha \leq 1$, let $BC^{0,\alpha}(V)$ be the Banach space of functions $\phi \in BC(V)$ which are uniformly Hölder continuous with exponent α. Its norm is defined by $\|\phi\|_{0,\alpha,V} := \|\phi\|_{\infty,V} + \sup_{x,y \in V, x \neq y} [|\phi(x) - \phi(y)|/|x-y|^\alpha]$. Let $BC^1(V) := \{\phi \in BC(V) | \partial_j \phi \in BC(V), \ j = 1, 2\}$, with the norm $\|\phi\|_{1,V} := \|\phi\|_{\infty,V} + \|\partial_1 \phi\|_{\infty,V} + \|\partial_2 \phi\|_{\infty,V}$ and $BC^{1,\alpha}(V) := \{\phi \in BC^1(V) | \partial_j \phi \in BC^{0,\alpha}(V), \ j = 1, 2\}$, with the norm $\|\phi\|_{1,\alpha,V} := \|\phi\|_{\infty,V} + \|\partial_1 \phi\|_{0,\alpha,V} + \|\partial_2 \phi\|_{0,\alpha,V}$. In addition, for the nonnegative integer n, J_n and Y_n denote the Bessel function and the Neumann function of order n, respectively. $H_n^{(1)} := J_n + iY_n$ is Hankel function of the first kind of order n.

Theoretically, the key issue for the direct rough interface scattering problem is to establish the uniqueness. Few work has been done in recent years. Actually, the uniqueness of the problem (BVP) follows directly from the results in [3], where the authors studied a more general case: scattering by an infinite inhomogeneous conducting or dielectric layer at the interface between semi-infinite homogeneous dielectric half-spaces. Combination of assumptions on the variation of the index of refraction in the layer and the radiation condition will give the uniqueness of the problem. A related but different problem has been studied in [4], where a uniqueness theorem is proved for the scattering problem by an infinite rough interface with the so-called conductive interface transmission conditions provided that the medium below the interface is lossy. By the potential method, the problem is reduced to an equivalent system of integral equations and existence results can be established by the generalized Fredholm theory in [5,6].

As mentioned above, the direct scattering problem can be transformed to an equivalent system of integral equations, consisting of second-kind integral equations on the real line. Numerically, one of the suitable methods to solve the second-kind integral equation on finite intervals with logarithmically singular periodic kernels is the Nyström method which is first applied to the obstacle scattering problem [7]. Recently, in [8] the Nyström method was extended to the case for a class of