Computers and Mathematics with Applications 71 (2016) 1503–1511

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On semi-convergence of the generalized shift-splitting iteration method for singular nonsymmetric saddle point problems

Yang Cao [a,b], Shu-Xin Miao [c,*]

[a] School of Transportation, Nantong University, Nantong, 226019, China
[b] Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing 210023, China
[c] College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China

ARTICLE INFO

Article history:
Received 19 October 2015
Received in revised form 15 February 2016
Accepted 20 February 2016
Available online 14 March 2016

Keywords:
Singular nonsymmetric saddle point problem
Shift-splitting iteration
Semi-convergence
Pseudo-spectral radius

ABSTRACT

Recently, a class of generalized shift-splitting iteration methods were proposed for solving nonsingular nonsymmetric saddle point problems (Cao et al., 2015). In this paper, the generalized shift-splitting iteration method is extended to solve singular nonsymmetric saddle point problems. Semi-convergence of the generalized shift-splitting iteration method is carefully analyzed, which shows that the iterative sequence generated by the generalized shift-splitting iteration method converges to a solution of the singular saddle point problem unconditionally. Numerical results verify the robustness and efficiency of the generalized shift-splitting iteration method.

1. Introduction

Consider the iterative solution of large sparse nonsymmetric saddle point problems

$$\mathcal{A}u = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsymmetric positive definite, $B \in \mathbb{R}^{m\times n}$ $(m \le n)$ is a rectangular matrix of rank r, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given vectors. When $r = m$, the coefficient matrix is nonsingular and the nonsymmetric saddle point problem (1.1) has a unique solution. When $r < m$, the coefficient matrix is singular, and at the moment, we call (1.1) a singular nonsymmetric saddle point problem. Moreover, in such case, we suppose that the singular saddle point problem (1.1) is consistent, i.e., $b \in R(\mathcal{A})$, the range of $\mathcal{A}$. Both nonsingular and singular saddle point problems arise in many areas of scientific computing and engineering applications. We refer to [1] for an overview of its applications.

For its property of large and sparsity, the nonsymmetric saddle point problem (1.1) is suitable for being solved by the iterative method. A number of effective iterative methods have been proposed to solve saddle point problems, such as the Uzawa-type iterative methods [2–7], residual reduction algorithm [8], Krylov subspace methods [9,10], the Hermitian and skew-Hermitian splitting (HSS)-like iterative methods [11–13] and so on. See also [1] for a general introduction to the different solution methods. Most of these iterative methods can be used to solve both singular and nonsingular saddle point

* Corresponding author.
E-mail address: shuxinmiao@gmail.com (S.-X. Miao).

http://dx.doi.org/10.1016/j.camwa.2016.02.027
0898-1221/

problems [14–16]. Recently, for solving the nonsingular nonsymmetric saddle point problem (1.1), a class of generalized shift-splitting (GSS) iteration methods are proposed in [17]

$$\frac{1}{2}\begin{bmatrix}\alpha I + A & B^T \\ -B & \beta I\end{bmatrix}\begin{bmatrix}x_{k+1} \\ y_{k+1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\alpha I - A & -B^T \\ B & \beta I\end{bmatrix}\begin{bmatrix}x_k \\ y_k\end{bmatrix} + \begin{bmatrix}f \\ g\end{bmatrix}, \tag{1.2}$$

where α and β are two positive real parameters, and I is the identity matrix with appropriate dimension. The GSS iteration method is based on the following matrix splitting of the coefficient matrix $\mathcal{A}$

$$\mathcal{A} = \frac{1}{2}(\Omega + \mathcal{A}) - \frac{1}{2}(\Omega - \mathcal{A}), \quad \text{with } \Omega = \begin{bmatrix}\alpha I & 0 \\ 0 & \beta I\end{bmatrix}. \tag{1.3}$$

Theoretical analysis in [17] showed that the GSS iteration method converges to the unique solution of the nonsingular nonsymmetric saddle point problem. In fact, the GSS iteration method is a generalization of the shift-splitting iteration method, which was first proposed by Bai et al. [18] to solve a class of non-Hermitian positive definite linear systems. Then it was extended by Cao et al. [19,20] to solve standard saddle point problem where A is symmetric positive definite. The induced GSS preconditioners for symmetric and nonsymmetric saddle point matrices have been studied in [17,19,20]. Numerical results in [17–20] show that the shift-splitting iteration methods and the induced preconditioners are effective and robust.

In this paper, we shall apply the GSS iteration method to solve the singular nonsymmetric saddle point problem (1.1). The semi-convergent properties of the GSS iteration method are studied in detail. The remainder of the paper is organized as follows. In Section 2, the semi-convergence concepts and a useful lemma are given. In Section 3, the semi-convergence of the GSS iteration method is studied and the spectral properties of the induced GSS preconditioned saddle point matrix are obtained correspondingly. In Section 4, numerical experiments are performed to show the feasibility and effectiveness of the GSS iteration method and the GSS preconditioned Krylov subspace methods for solving the singular nonsymmetric saddle point problem.

2. Basic concept and lemma

Throughout this paper, A^T, $\sigma(A)$, $\rho(A)$, $null(A)$, $rank(A)$ and $index(A)$ denote the transpose, the spectral set, the spectral radius, the null space, the rank and the index of the matrix A, respectively. Let

$$\mathcal{M} = \frac{1}{2}\begin{bmatrix}\alpha I + A & B^T \\ -B & \beta I\end{bmatrix}, \qquad \mathcal{N} = \frac{1}{2}\begin{bmatrix}\alpha I - A & -B^T \\ B & \beta I\end{bmatrix},$$

then the splitting (1.3) becomes

$$\mathcal{A} = \mathcal{M} - \mathcal{N}$$

and the iteration scheme (1.2) can be rewritten as

$$\begin{bmatrix}x_{k+1} \\ y_{k+1}\end{bmatrix} = \Gamma \begin{bmatrix}x_k \\ y_k\end{bmatrix} + c, \tag{2.1}$$

where

$$\Gamma = \mathcal{M}^{-1}\mathcal{N} = \begin{bmatrix}\alpha I + A & B^T \\ -B & \beta I\end{bmatrix}^{-1}\begin{bmatrix}\alpha I - A & -B^T \\ B & \beta I\end{bmatrix} \tag{2.2}$$

is the iteration matrix of the GSS iteration method and $c = \mathcal{M}^{-1}\begin{bmatrix}f \\ g\end{bmatrix}$. Any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for the Krylov subspace methods. The splitting preconditioner corresponds to the generalized shift-splitting iteration (1.2) or (2.1) is the matrix $\mathcal{M}$, which we call the generalized shift-splitting (GSS) preconditioner for the singular nonsymmetric saddle point matrix $\mathcal{A}$.

To implement the GSS iteration method or apply the GSS preconditioner to a Krylov subspace method efficiently, we need to solve a system of linear equations with the coefficient matrix $\mathcal{M}$ at each iteration step. Note that the Algorithm 2.1 presented in [17] is also suitable to implement the GSS iteration method or the GSS preconditioner when it is used to solve the singular saddle point problem (1.1). According to [17, Algorithm 2.1], a system of linear equation with the coefficient matrix $\alpha I + A + \frac{1}{\beta}B^T B$ is needed to solve at each iteration step.

When $\mathcal{A}$ is nonsingular, for any initial vector $[x_0^T, y_0^T]^T$ the iteration scheme (2.1) converges to the exact solution of (1.1) if and only if $\rho(\Gamma) < 1$. But for the singular matrix $\mathcal{A}$, we have $1 \in \sigma(\Gamma)$ and $\rho(\Gamma) \geq 1$, so that one can require only the *semi-convergence* of the iteration scheme (2.1), see [21,22].