Computers and Mathematics with Applications 71 (2016) 1524–1536

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Ground state solutions for Kirchhoff type equations with asymptotically 4-linear nonlinearity

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ARTICLE INFO

Article history:
Received 14 September 2015
Received in revised form 9 January 2016
Accepted 24 February 2016
Available online 16 March 2016

Keywords:
Kirchhoff type equations
Variational method
Asymptotically 4-linear
Nehari manifold
Ground state

ABSTRACT

This paper is concerned with the following Kirchhoff type equation:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla v|^2\right)\Delta v = f(x,v) & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega. \end{cases}$$

Assuming that the primitive of f is asymptotically 4-linear as $|v| \to \infty$, a homeomorphism between the Nehari manifold and a subset of unit sphere is constructed based on some observations and new techniques. Inspired by recent work of Szulkin and Weth (2010), ground state solutions for the equation above are obtained, as well as infinitely many pairs of nontrivial solutions for case $f(x, v)$ is odd with respect to v.

1. Introduction and main results

Consider the following Kirchhoff type equation:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla v|^2\right)\Delta v = f(x,v) & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ($N = 1, 2,$ or 3) and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Problem (1.1) has been widely investigated in the literature over the past several decades, especially on the existence of positive solutions, sign-changing solutions, ground states and multiple solutions, see for example, [1–10] and the references therein.

This problem is related to the stationary analogue of the equations

$$v_{tt} - \left(a+b\int_{\Omega}|\nabla v|^2\right)\Delta v = f(x,v), \tag{1.2}$$

where v denotes the displacement, $f(x, v)$ the external force and b the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. Equations of this type were first proposed by Kirchhoff [11] as an extension of the classical d'Alembert wave equations for free vibrations of elastic strings. Such problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω, and the first equation in (1.1) is no longer a

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http://dx.doi.org/10.1016/j.camwa.2016.02.037
0898-1221/

pointwise equality. Such a phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. Nonlocal effect also finds its applications in some biological systems, a parabolic version of Eq. (1.1) can, in theory, be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed dependent on the energy of the entire system with v being its population density. For more mathematical and physical background of problem (1.1), we refer the readers to papers [3,4,11–13] and the references therein.

There has been increasing interest in studying problem (1.1) by variational methods after the pioneering work of Lions [14]. Corrêa [4] studied (1.1) by taking advantage of fixed point theorems and obtained positive solutions for both $N = 1$ and $N \geq 2$. Based on the fountain theorems, existence of infinitely many solutions for (1.1) was established by He and Zou [5] under classical condition (AR) or the following monotonic condition:

(F0) $t \mapsto \frac{f(x,t)}{|t|^3}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

In [6], multiplicity results were also proved by them for the case that f having an oscillating behavior. Under (F0) and some subcritical assumptions on autonomous f, least energy sign-changing solutions for problems (1.1) were obtained recently by Wei [9] with the aid of quantitative deformation lemma and degree theory. For problem (1.1) with a critical term, nontrivial solution of mountain pass type was proved by Alves et al. [1], see also recent paper [7] where two positive solutions for (1.1) were established via the perturbation methods. In [8], Perera and Zhang considered the asymptotically 4-linear case, and the assumptions for the asymptotic behaviors of f near zero and infinity are

$$\lim_{t\to 0} \frac{f(x,t)}{at} = \lambda, \qquad \lim_{t\to\infty} \frac{f(x,t)}{bt^3} = \mu, \quad \text{uniformly in } x, \tag{1.3}$$

where $\lambda, \mu \in \mathbb{R}$. Particularly, they obtained nontrivial solutions for (1.1) with the aid of Yang index and critical group. For the 4-sublinear, 4-superlinear and asymptotically 4-linear case, positive, negative and sign-changing solutions were both obtained by them in [10] by using invariant sets of descent flow. Similar results can be found in [15] for 4-superlinear case, and in [16] for asymptotically 4-linear case provided that (F0) and $f(x, t) \equiv 0$ for all $t \leq 0$ and $x \in \overline{\Omega}$ are satisfied. Recently, Chen et al. [3] investigated (1.1) by using the Nehari manifold approach for case that

$$f(x,t) = \lambda h(x)|t|^{q-2}t + g(x)|t|^{p-2}t, \quad 1 < q < 2 < p < 2^*,$$

where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$, and $h, g \in C(\overline{\Omega})$ satisfy

$$h^+ := \max\{h, 0\} \neq 0, \quad \text{and} \quad g^+ := \max\{g, 0\} \neq 0. \tag{1.4}$$

Some existence results were obtained there both for case $p > 4$, $p = 4$ and $p < 4$. In recent paper [2], the Nehari manifold method was also applied to 4-superlinear equation (1.1), moreover, ground state solution and infinitely many solutions were obtained under a stronger version of (F0), i.e.,

(F0′) $t \mapsto \frac{f(x,t)}{|t|^3}$ is positive for $t \neq 0$, and nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

The key point to use the method lies on showing that the functional corresponding to (1.1) has a unique maximum point along the direction of nontrivial v, then one may establish a homeomorphism between the Nehari manifold $\mathcal{N}$ and the unit sphere S. Therefore, looking for a ground state solution for (1.1) could be reduced into that of considering a minimizing sequence on Nehari manifold. Problem (1.1) with asymptotically linear f was considered in [17], and existence of positive solutions was obtained there. For similar Kirchhoff's type problem in $\mathbb{R}^N$ including the semiclassical states results, we refer the readers to [18–26] and the references therein.

Motivated by the above works, we are going to consider (1.1) for the asymptotically 4-linear case with more general nonlinearity and establish the existence of ground state solutions, as well as infinitely many solutions. As we know, results of the problem (1.1) with four times growth are few since there is no mountain pass structure and the standard variational methods cannot be used directly on Nehari manifold in order to extract the critical points of the functional. On the other hand, the absence of higher-order term of the nonlinearity and the competing effect of the nonlocal term $\left(\int_\Omega |\nabla v|^2\right) \Delta v$ with the nonlinearity prevent us from using the usual method of Nehari manifold [27]. So some techniques or new methods are looked forward to being introduced which is the right issue this paper intends to address.

Let $E := H_0^1(\Omega)$. The natural space for the Kirchhoff type equation (1.1) is the Sobolev space E equipped with the inner product

$$(v, w) = \int_\Omega \nabla v \cdot \nabla w \mathrm{d}x, \quad \forall\, v, w \in E, \tag{1.5}$$

and the norm $\|v\| = (v, v)^{1/2}$. It is well known that E is continuously embedded in $L^s(\Omega)$ for $s \in [1, 2^*]$, and compactly for $s \in [1, 2^*)$ (see [28, Theorem 1.9]). Then there exists $\gamma_s > 0$ such that

$$\|v\|_s \leq \gamma_s \|v\|, \quad \forall\, v \in E, \tag{1.6}$$