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On asymptotic summations of almost diagonal difference systems and their applications

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ABSTRACT

In this paper, we derive some new results of asymptotic summations of left and right almost diagonal difference systems. In particular, we study a special case where the diagonal elements are constants with magnitude 1. Conditions on the perturbation matrices are given under which the system is left and right almost diagonal. An application of this result in a discrete adiabatic oscillator equation is discussed. We also introduce a result for the case when the diagonal matrix is approaching the identity matrix I. An example is given to illustrate that the new result can be applied to some cases that do not satisfy the "Growth Condition".

1. Introduction

Consider the matrix difference equation

$$Y(t+1) = A(t)Y(t), \quad t \geq a \tag{1}$$

where $t \in \mathbb{Z}$ and $A(t), Y(t)$ are $n \times n$ matrix functions. It is known that this system has no general explicit solution. Let $A(t) = D(t) + R(t)$ in which $D(t)$ is a diagonal matrix and $R(t)$ is a perturbation matrix function. Throughout this paper we adopt the following notation.

$$D(t) = \text{diag}\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}, \tag{2}$$

$$R(t) = \left\{r_{jk}(t)\right\}_{j,k=1}^{n}. \tag{3}$$

Thus we rewrite the difference system (1) as

$$Y(t+1) = (D(t) + R(t))\, Y(t). \tag{4}$$

The solution of the diagonal difference system $Y(t+1) = D(t)Y(t),\ t \geq a$, with the initial condition $Y(a) = I$ is given by

$$\Phi(t) = \left[\prod_{l=a}^{t-1} D(l)\right]. \tag{5}$$

Many studies have been done to seek conditions on $D(t)$ and $R(t)$ such that (4) has a fundamental matrix asymptotically close to $\Phi(t)$. Let us recall the definitions of left and right almost diagonal difference systems given in [1,2].

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Definition 1. The system (4) is *right almost diagonal* if it possesses an asymptotic representation

$$Y(t) = \Phi(t)(I + P(t)), \tag{6}$$

with $P(t) \to 0$ as $t \to \infty$. Similarly, if representation

$$Y(t) = (I + Q(t))\Phi(t) \tag{7}$$

holds and $Q(t) \to 0$ as $t \to \infty$, the system (4) will be called *left almost diagonal*.

Most studies have discussed only left almost diagonal systems, see [2–7]. That was because the well known Levinson's perturbation theorem for differential systems was on left almost diagonal systems. Also see [8] for the results in a wider setting of time scales which included both differential and difference systems. Among them, a widely applied result is given by Benzaid and Lutz in [3]. It is the discrete analog of Levinson's perturbation theorem. It states that the system (4) is left almost diagonal if $D(t)$ satisfies the "dichotomy conditions": for each pair of integers $j \neq k$ and $t > a$, either

$$\prod_{l=a}^{t} \frac{\lambda_j(l)}{\lambda_k(l)} \to \infty \quad \text{as } t \to \infty, \quad \text{and} \quad \prod_{l=t_1}^{t_2} \frac{\lambda_j(l)}{\lambda_k(l)} \geq \mu > 0 \quad \text{for all } a \leq t_1 \leq t_2, \tag{8}$$

or

$$\prod_{l=t_1}^{t_2} \frac{\lambda_j(l)}{\lambda_k(l)} \leq K \quad \text{for all } a \leq t_1 \leq t_2; \tag{9}$$

and $R(t)$ satisfies the "growth condition", i.e.,

$$\frac{1}{\lambda_j(t)} R(t) \in l^1. \tag{10}$$

In [1], some results on right almost diagonal systems for the special case of a potentially oscillatory system were obtained. In these results, it was not necessary for $R(t)$ to satisfy the growth condition (10).

Definition 2. A difference system (4) is called potentially oscillatory on $[a, \infty)$ if there exist positive numbers M_1 and M_2 with $M_2 < M_1$ such that

$$0 < M_2 \leq \left| \prod_{l=t_1}^{t_2} \frac{\lambda_k(l)}{\lambda_j(l)} \right| \leq M_1, \quad j, k = 1, 2, \ldots, n, \ j \neq k, \tag{11}$$

for all $a \leq t_1 < t_2 < \infty$.

However, in the main result of [1] (see Theorem 2 in sequel), the conditions are mainly on the matrix function $\tilde{R}(t)$, which includes the elements from both $D(t)$ and $R(t)$ (see its definition in (12)). Therefore it is not very easy to apply Theorem 2.

Some more general results of sufficient conditions under which a difference system has "linear asymptotic equilibrium" could be found in [9,10].

So far the relationship between left and right almost diagonal systems has not been studied. In this paper, we will first discuss the relationship of left and right almost diagonal systems for potentially oscillatory systems. Then we obtain a new result for a special case when the diagonal elements are constant with magnitude 1. Here we use the perturbation matrix $R(t)$, rather than $\tilde{R}(t)$, to check whether a system is almost diagonal. This result is applied to an adiabatic oscillator equation at the end of the paper. Moreover, we derive a theorem that deals with the case when $D(t)$ approaches I, and give an example to show its application.

2. Asymptotic summations of potentially oscillatory systems

The following theorem shows that for a potentially oscillatory system, a left almost diagonal system is equivalent to a right almost diagonal system.

Theorem 1. *A potentially oscillatory system (4) is right almost diagonal if and only if it is left almost diagonal.*

Proof. Assume that the system (4) is left almost diagonal. Then there exists a matrix $Q(t)$ with $Q(t) \to 0$ such that the system (4) has a fundamental matrix $Y(t)$ in the form $Y(t) = (I + Q(t))\Phi(t)$. For that $Q(t)$, let

$$P(t) = \Phi(t)^{-1}(I + Q(t))\Phi(t) - I.$$

Then we can obtain $\Phi(t)(I + P(t)) = (I + Q(t))\Phi(t)$, which means $\Phi(t)(I + P(t))$ is a fundamental matrix for Eq. (4). We only need to show that $P(t) \to 0$ as $t \to \infty$.