Computers and Mathematics with Applications 66 (2013) 2211–2219

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Orbit distributions of iterated function systems with finitely many forms

Richard E. Lampe *

Department of Mathematics and Science, J. Sargeant Reynolds Community College, P.O. Box 85622, Richmond, VA 23285-5622, United States

ARTICLE INFO

Keywords:
Iterated function system
Distribution
Discrete dynamics

ABSTRACT

Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a finite family of measure preserving self maps on a complete measure space (X, Σ, μ), indexed by the set I. For a sequence $\alpha = a_1, a_2, \ldots$, where $a_i \in I$ the n-fold composition with respect to α is $F_\alpha^n = f_{a_n} \circ F_\alpha^{n-1}$. When the n-fold compositions from the family $\mathcal{F}$ take finitely many forms, the discrete time distribution of the orbit of $F_\alpha^k(x_0)$ is a weighted average of the discrete time distributions of the orbits of the finite forms at the point x_0 for μ-almost all x_0 and for almost all sequences α. The weighted average is arrived at by showing that an independence condition holds through an application of the strong law of large numbers to a subsequence of the Rademacher functions. When the discrete time distributions of the finite forms are identical for μ-almost all $x_0 \in X$ the weighted sum of the discrete time distributions reduces to the single valued distribution for any one of the finite forms.

1. Introduction, background and examples

Much emphasis concerning iterated function systems has been placed on stable, forward or backward iterative processes often used to identify fixed fractal attractors. In that setting, iterated function systems have been variously defined as finite collections of self maps on a set both with and without conditions on the maps themselves. Hutchinson and John [1,2] defined an iterated function system by requiring the self-maps to be contractive, assuring convergence. More general weak contractivity conditions, including average contraction conditions have also been developed [3–5]. In a motivating example, Diaconis and Freedman considered a random system and both backward, stable iteration that converges to a fractal attractor as well as forward unstable iteration that proceeds ergodically through the state space [4]. In this paper we focus on the forward multiple-valued iteration of functions from a finite collection of measure preserving self-maps on a set and the subsequent identification of their resulting orbit distributions. In contrast with the contractive mapping case where the iterating function at each step is generally taken to be randomly selected according to a distribution, we take a more deterministic approach. Multiple-valued forward iteration proceeds with respect to a fixed branch sequence that determines the outer composing function at each step [6,7]. Our results are then measure-theoretic in nature: we show that time averages equal space averages for almost all starting points and for almost all branch sequences.

In Section 1 we provide some background, introduce preliminary definitions and provide several examples that illustrate the iterated function system setting. In Section 2 we illustrate the main idea that the finite form conditions lead to weighted averages of orbit distribution functions. In Section 3 we introduce the Rademacher functions and prove that they are stochastically independent (see Fig. 1). Using this, we prove the two main results: for both finite form conditions time averages equal space averages for the multiple-valued orbit of almost every starting point for almost every branch sequence. We conclude with Section 4 where we summarize the results and apply them to the examples introduced in Section 1.

* Tel.: +1 804 852 9275.
E-mail addresses: rlampe@reynolds.edu, rlampe@richmond.edu.

0898-1221/$ – see front matter
http://dx.doi.org/10.1016/j.camwa.2013.06.021

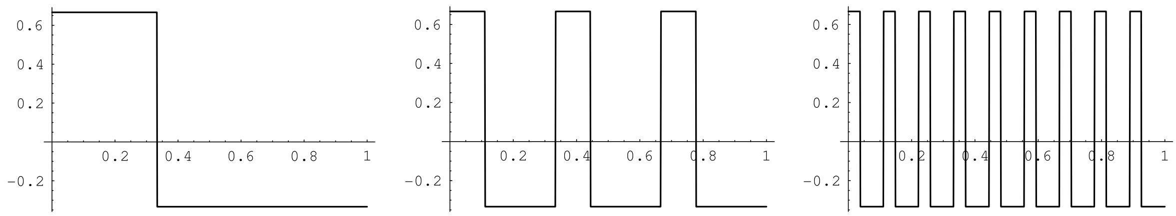

Fig. 1. The first three Rademacher functions for $j \equiv 0$ modulo 3.

We begin with a definition.

Definition 1. Let (X, Σ, μ) be a complete measure space and let $\mathcal{F} = \{f_i\}_{i \in I}$ be a family of self maps on X, indexed by the finite set $I = \{0, 1, \ldots, n-1\}$. For $x_0 \in X$ and for a sequence $\alpha = a_1, a_2, \ldots$ with $a_i \in I$, the k^{th} iterate of x_0 with respect to α is

$$F_\alpha^k(x_0) = f_{a_k} \circ F_\alpha^{k-1}(x_0)$$

for each $k \in \mathbb{N}$. The pair $(\mathcal{F}, X)$ is an iterated function system.

We call the functions $f_i \in \mathcal{F}$ the branches of the iterated function system, and the sequence α used to determine the composing function at each step of iteration is called a branch sequence. When the branches of the iterated function system are measure preserving maps, we will call the iterated function system a measure-preserving iterated function system.

We suppose that the functions $f_i \in \mathcal{F}$ are related in either one of the following two senses:

1. There is a finite family of self-maps $\mathcal{G} = \{g_0, g_1, \ldots, g_{m-1}\}$ with the property that for each sequence α, for each $x_0 \in X$, and for each $k \in \mathbb{N}$, there exists a unique j, $0 \le j < m$, such that

$$F_\alpha^k(x_0) = g_j^k(x_0). \tag{1}$$

2. There is a finite collection of points $\mathcal{H} = \{y_0, y_1, \ldots, y_{m-1}\}$ and a fixed function $f_i \in \mathcal{F}$, such that for each $x_0 \in X$, and for each $k \in \mathbb{N}$, there exists a unique j, $0 \le j < m$, such that

$$F_\alpha^k(x_0) = f_i^k(y_j). \tag{2}$$

In each case, j depends on α, k and x_0. We refer to these as finite form conditions.

We present here several examples and generalizations of iterated function systems that satisfy finite form condition (1) or (2).

1. Let $\mathcal{F} = \{f_0 = T(x), f_1 = 1 - T(x)\}$ where $T(x)$ is the tent map on $X = [0, 1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2 - 2x & \text{otherwise.} \end{cases}$$

Because $T(1-x) = T(x)$ for all $x \in [0, 1]$, it follows that the kth multiple valued iterate takes one of two forms

$$F_\alpha^k(x_0) = \begin{cases} f_0^k(x_0) & \text{if } a_k = 0 \\ f_1^k(x_0) & \text{if } a_k = 1. \end{cases}$$

The a_k are the terms of the branch sequence α introduced in Definition 1. The forms of the multiple valued iterates in this case are a finitely generated infinite semi-group. In general, this need not be the case. For example, the iterates could take the form of an infinite free semi-group, in which case identification of discrete time distributions would prove more difficult, if not impossible. This example is a measure preserving iterated function system satisfying finite form condition (1). In the present case, $\mathcal{G} = \mathcal{F}$. □

2. Let $\mathcal{F} = \{f_0 = B(x), f_1 = 1 - B(x)\}$ where $B(x)$ is the base 2 shift on $X = [0, 1]$ defined by

$$B(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2x - 1 & \text{otherwise.} \end{cases}$$

Note that $B(1-x) = 1 - B(x)$ for all $x \in [0, 1]$. Moreover, $B(1 - B(x)) = B^2(1-x)$ and $1 - B(1 - B(x)) = B^2(x)$. An induction argument shows that for each fixed k it is either the case that $F_\alpha^k(x_0) = B^k(x_0)$ or $F_\alpha^k(x_0) = B^k(1-x_0)$. The form of $F_\alpha^k(x_0)$ is determined by the number of occurrences of the function $f_1 = 1 - B(x)$ used in the composition, in other