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Solutions to the 2-dimensional isothermal Euler–Poisson equations with a cosmological constant

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ABSTRACT

In this paper, we study the Euler–Poisson equations describing the evolution of the gaseous star with a negative cosmological constant. We construct explicit solutions for the isothermal case in $\mathbb{R}^2$, which contain blowup solutions and the solutions with time periodicity. Furthermore, the blowup rate of the explosive solutions is also given.

1. Introduction

Let $\rho = \rho(t, x)$ and $\mathbf{u} = \mathbf{u}(t, x)$ be the density and the velocity respectively, then the isentropic Euler–Poisson equations describing a self-gravitating fluid can be formulated in the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = -\rho \nabla \Phi, \\ \Delta \Phi = \alpha(N)(\rho - \Lambda). \end{cases} \tag{1.1}$$

In system (1.1), P is the pressure satisfying

$$P = K\rho^\gamma = \frac{\rho^\gamma}{\gamma},$$

where $\gamma \geq 1$ is the adiabatic exponent. In particular, the fluid is called isothermal if $\gamma = 1$. Λ denotes the cosmological constant. When Λ is positive, the space is open; while it is negative, the space is closed; and when it is zero, the space is flat. $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 1,\ \alpha(2) = 2\pi$ and

$$\alpha(N) = N(N-2)\text{vol}(B), \quad N \geq 3,$$

where $\text{vol}(B) = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ is the volume of the unit ball in $\mathbb{R}^N$. In the third equation of (1.1), $\Phi(t, \mathbf{x})$ is the self-gravitational potential field which can be solved as

$$\Phi(t, \mathbf{x}) = \int_{\mathbb{R}^N} G(\mathbf{x} - \mathbf{y})(\rho(t, \mathbf{y}) - \Lambda) d\mathbf{y},$$

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where G is Green's function for the Poisson equation in the N-dimensional spaces defined by

$$G(\mathbf{x}) := \begin{cases} |\mathbf{x}|, & N = 1, \\ \ln|\mathbf{x}|, & N = 2, \\ -\dfrac{1}{|\mathbf{x}|^{N-2}}, & N \geq 3. \end{cases}$$

Regarding the background knowledge of the Euler–Poisson equations with a cosmological constant Λ, we refer to [1,2]. When $N = 3$, system (1.1) is the classical (non-relativistic) descriptions of a galaxy in astrophysics, see [3,4] for the details about the system. We also refer to [5–8] for the blowup criteria and stability results of some related Euler–Poisson system.

Let us recall some known results for the system (1.1). In the case of $\Lambda = 0$, Goldreich–Weber in [9] constructed analytical blowup solutions of the 3-dimensional Euler–Poisson equations for the non-rotating gas spheres with $\gamma = 4/3$. When $N \geq 3$ and $\gamma = (2N-2)/N$, Deng–Xiang–Yang in [10] established the existence of the blowup solutions. Analytical blowup solutions in $\mathbb{R}^2$ were constructed by Yuen [11] with $\gamma = 1$. Periodic solutions to the 3-dimensional Euler–Poisson equations with a negative cosmological constant were first constructed by Yuen [12] in 2009. Inspired by the work of Yuen, in this paper, we obtain some explicit solutions to the 2-dimensional Euler–Poisson equations with a negative cosmological constant.

Here, we are concerned on the solutions with spherical symmetry

$$\rho(t, \mathbf{x}) = \rho(t, r), \qquad \mathbf{u} = \frac{\mathbf{x}}{r}V(t, r) =: \frac{\mathbf{x}}{r}V, \tag{1.2}$$

where $r = |\mathbf{x}| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$. Then the Poisson equation $(1.1)_3$ is transformed into

$$r^{N-1}\Phi_{rr} + (N-1)r^{N-2}\Phi_r = \alpha(N)(\rho - \Lambda)r^{N-1}. \tag{1.3}$$

Integrating on both sides of (1.3) gives

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r (\rho(t, s) - \Lambda)s^{N-1} ds. \tag{1.4}$$

From now on, we set $N = 2$. In the case of spherical symmetry, system (1.1) can be reduced to the following form

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{1}{r}\rho V = 0, \\ \rho(V_t + VV_r) + K\rho_r = -\dfrac{2\pi\rho}{r} \displaystyle\int_0^r (\rho(t, s) - \Lambda)s ds. \end{cases} \tag{1.5}$$

Now, we state our main results of the paper.

Theorem 1.1. *Let $\gamma = 1$ and $\Lambda < 0$, then there exist a set of solutions for the 2-dimensional Euler–Poisson equations (1.5), which can be expressed as below*

$$\begin{cases} \rho(t, r) = \dfrac{1}{a^2(t)} e^{y(r/a(t))}, \qquad V(t, r) = \dfrac{\dot{a}(t)}{a(t)} r, \\ \ddot{a}(t) = \dfrac{-\lambda}{a(t)} + \pi\Lambda a(t), \qquad a(0) = a_0 > 0, \qquad \dot{a}(0) = a_1, \\ \ddot{y}(z) + \dfrac{1}{z}\dot{y}(z) + \dfrac{2\pi}{K} e^{y(z)} = \mu, \qquad y(0) = \alpha, \qquad \dot{y}(0) = 0, \end{cases} \tag{1.6}$$

where $K > 0$, μ, α and λ are constants such that $\mu = 2\lambda/K$ and λ is sufficiently small. Moreover, we have

(1) *When $\lambda \geq 0$, the solutions blow up at a finite time T.*
(2) *When $\lambda < 0$, the solutions are time periodic. Especially, when $a_0 = \sqrt{-\lambda}$ and $a_1 = 0$, $a(t) \equiv \sqrt{-\lambda}$.*

When $N = 3$, the explicit blowup solutions were obtained under the conditions $a_1 \leqslant 0$ (see [12]). In this work, we give the proof without this condition. When $\lambda < 0$, the solutions are periodic. Compared to the time-periodic solutions obtained by Yuen [12] with $\gamma = 4/3$ in 3d, the form of the density $\rho(t, r)$ is different from 3d case. Moreover, we here use an elementary analysis technique to prove the periodic property, which is slightly different from the work of Yuen.

Next, we consider the blowup rate of solutions to (1.5) when $\lambda > 0$. In Theorem 1.2, we will discuss when time t tends to the critical time T, how fast the blowup solution tends to infinity.