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Symmetry analysis and reductions of the two-dimensional generalized Benney system via geometric approach

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ABSTRACT

In this work, the symmetry group and similarity reductions of the two-dimensional generalized Benney system are investigated by means of the geometric approach of an invariance group, which is equivalent to the classical Lie symmetry method. Firstly, the vector field associated with the Lie group of transformation is obtained. Then the point transformations are proposed, which keep the solutions of the generalized Benney system invariant. Finally, the symmetry reductions and explicitly exact solutions of the generalized Benney system are derived by solving the corresponding symmetry equations.

1. Introduction

Lie symmetry analysis method for differential equations was originally developed by Sophus Lie [1] and is a highly algorithmic approach to find the symmetry group and reduction of various differential equations both integrable and nonintegrable. In recent years, there have been considerable important advances in this method including the nonclassical method of Bluman and Cole [2], Lie–Bäcklund symmetry [3], potential symmetry [4,5], geometric approach of invariance groups [6], symmetries of differential–difference equations [7,8], Clarkson and Kruskal (CK) direct method [9], modified CK direct method [10], and so on [11,12]. Among these developments, the geometric approach of invariance groups proposed by Harrison and Estabrook [6] has been based on Cartan's formulation of differential equations in terms of exterior differential forms [13] and can be easily applied to various differential equations [14–17]. This approach consists of determining isovector fields of a closed ideal of certain exterior differential forms defined on a properly extended manifold. The isovector fields are determined by vector fields whose orbits generate transformations under which the ideal remains invariant.

The Benney system [18] was first derived by Benney from the two-dimensional and time-dependent motion of an inviscid homogeneous fluid in a gravitational field by assuming the depth of the fluid to be small compared to the horizontal wavelengths. In practice, the Benney system is used in various representations. In the past years, much work focused on symmetry group analysis of the Benney system in the form of the nonlinear integro-differential equations [19]. Moreover, some similarity reductions and similarity solutions of the Benney system in integro-differential form are proposed.

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Another form of the Benney system presented by Zakharov [20] is the generalized Benney system

$$\begin{cases} a_t + av_x + va_x = 0, \\ v_t + vv_x + w_x = 0, \\ w_y + a_x = 0, \end{cases} \tag{1}$$

which is a certain two-dimensional generalization of one-dimensional gas dynamic equation [20,21]. This equation is a dispersionless limit of the following coupled equation

$$\begin{cases} a_t + av_x + va_x = -\frac{1}{2}a_{xx}, \\ v_t + vv_x + w_x = \frac{1}{2}v_{xx}, \\ w_y + a_x = 0, \end{cases} \tag{2}$$

by the limiting procedure [20]

$$\frac{\partial}{\partial t} \to \epsilon \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial x} \to \epsilon \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y} \to \epsilon \frac{\partial}{\partial y}, \quad \epsilon \to 0. \tag{3}$$

It is remarked that the coupled equation (2) is integrable and is a compatibility condition for the linear equation

$$\begin{cases} \psi_{xy} = v\psi_y + a\psi, \\ \psi_t = -\frac{1}{2}\psi_{xx} - w\psi. \end{cases} \tag{4}$$

To our knowledge, there are no studies on the Lie symmetry analysis and reductions of the two-dimensional generalized Benney system (1). Thus the main purpose of the present work is to study the symmetry group properties and similarity reductions of this system by geometric approach of the invariance groups [6].

2. Symmetry analysis via geometric approach

In order to apply the geometric approach [6] to the generalized Benney system (1), we rewrite system (1) in the language of exterior differential forms as

$$\begin{cases} \alpha_1 = da \wedge dx \wedge dy - adv \wedge dt \wedge dy - vda \wedge dt \wedge dy, \\ \alpha_2 = dv \wedge dx \wedge dy - vdv \wedge dt \wedge dy - dw \wedge dt \wedge dy, \\ \alpha_3 = dw \wedge dt \wedge dx - da \wedge dt \wedge dy, \end{cases} \tag{5}$$

where $\wedge$ is the exterior product of the exterior forms. Furthermore, it is seen that the differential form set $I = \{\alpha_1, \alpha_2, \alpha_3\}$, a fundamental ideal of the algebra of exterior forms $\wedge(M)$, where M is a manifold of dimension 6 of variables t, x, y, a, v and w, is closed. The set I is involution about variables t, x, y and gives back the original generalized Benney system (1) if we impose independence of these variables.

Define a vector field V over the manifold M with components $V^t, V^x, V^y, V^a, V^v, V^w$, in the direction of t, x, y, a, v and w, respectively; that is,

$$V = V^t\frac{\partial}{\partial t} + V^x\frac{\partial}{\partial x} + V^y\frac{\partial}{\partial y} + V^a\frac{\partial}{\partial a} + V^v\frac{\partial}{\partial v} + V^w\frac{\partial}{\partial w}, \tag{6}$$

where $V^t, V^x, V^y, V^a, V^v, V^w$ are functions of (t, x, y, a, v, w) to be determined.

According to the geometric approach [6], a vectors field V is said to be an isovector field if

$$L_V I(\alpha_1, \alpha_2, \alpha_3) \subset I(\alpha_1, \alpha_2, \alpha_3), \tag{7}$$

where $L_V(\cdot)$ is the Lie derivative of $(\cdot)$ over the vector field V, which requires

$$\begin{cases} L_V\alpha_1 = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3, \\ L_V\alpha_2 = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3, \\ L_V\alpha_3 = \mu_1\alpha_1 + \mu_2\alpha_2 + \mu_3\alpha_3, \end{cases} \tag{8}$$

where $\lambda_i, \xi_i, \mu_i (i = 1, 2, 3)$ are functions of variables (t, x, y, a, v, w).

By using the properties of Lie derivative, Eq. (8) can be rewritten as

$$\begin{aligned} &dV^a \wedge dx \wedge dy + da \wedge dV^x \wedge dy + da \wedge dx \wedge dV^y - V^a dv \wedge dt \wedge dy - adV^v \wedge dt \wedge dy \\ &\quad - adv \wedge dV^t \wedge dy - adv \wedge dt \wedge dV^y - V^v da \wedge dt \wedge dy - vda \wedge dt \wedge dV^y - vda \wedge dV^t \wedge dy \\ &= \lambda_1(da \wedge dx \wedge dy - adv \wedge dt \wedge dy - vda \wedge dt \wedge dy) + \lambda_2(dv \wedge dx \wedge dy \end{aligned}$$