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Global existence and energy decay result for a weak viscoelastic wave equations with a dynamic boundary and nonlinear delay term

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ABSTRACT

In this paper, we consider the weak viscoelastic wave equation

$$u_{tt} - \Delta u + \delta\Delta u_t - \sigma(t)\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u$$

with dynamic boundary conditions, and nonlinear delay term. First, we prove a local existence theorem by using the Faedo–Galerkin approximations combined with a contraction mapping theorem. Secondly, we show that, under suitable conditions on the initial data and the relaxation function, the solution exists globally in time, in using the concept of stable sets. Finally, by exploiting the perturbed Lyapunov functionals, we extend and improve the previous result from Gerbi and Said-Houari (2011).

1. Introduction

In this paper, we consider the following wave equation with dynamic boundary conditions and nonlinear delay term:

$$\begin{cases} u_{tt} - \Delta u + \delta\Delta u_t - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, \quad \text{in } \Omega \times (0,+\infty), \\ u = 0, \quad \text{on } \Gamma_0 \times (0,+\infty), \\ u_{tt} = -a\left[\dfrac{\partial u}{\partial \upsilon}(x,t) + \delta\dfrac{\partial u_t}{\partial \upsilon}(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta u(s)\dfrac{\partial u}{\partial \upsilon}(x,s)ds \right. \\ \quad \left. +\mu_1|u_t|^{m-1}u_t(x,t) + \mu_2|u_t(x,t-\tau)|^{m-1}u_t(x,t-\tau)\right], \quad \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad \text{on } \Gamma_1 \times (0,+\infty) \end{cases} \tag{1}$$

where $u = u(x,t), t \geq 0, x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of R^N, $(N \geq 1)$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \upsilon}$ denotes the unit outer normal derivative, μ_1 and μ_2 are positive constants. Moreover, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be precised later. This type of problem arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the

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strain rate. See [1]. This type of problem without delay (i.e $\mu_i = 0$), has been considered by many authors during the past decades and many results have been obtained (see [2–9]).

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass, (see [10–12] for more details). Which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (1) appear when we assume that is an exterior domain of R^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [13] for more details). Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen [2,3] in which the authors have made contributions to this field and in [5] the authors have studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \delta \Delta u_t = |u|^{p-1}u, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\left[\dfrac{\partial u}{\partial \upsilon}(x,t) + \delta \dfrac{\partial u_t}{\partial \upsilon}(x,t) + \alpha |u_t|^{m-1}u_t(x,t)\right], & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2}$$

and they have obtained several results concerning local existence which extended to the global existence by using the concept of stable sets, the authors have obtained also the energy decay and the blow up of the solutions for positive initial energy.

It is widely known that delay effects, which arise in many practical problems, source of some instabilities, in this way Datko and Nicaise [14–17] showed that a small delay in a boundary control turns a well-behaved hyperbolic system into a wild one which in turn, becomes a source of instability, where they proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1. \tag{3}$$

Recently, inspired by the works of Nicaise [14], Sthéphane Gherbi and B. Said-Houari [18] considered the following problem in bounded domain:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\left[\dfrac{\partial u}{\partial \upsilon}(x,t) + \alpha \dfrac{\partial u_t}{\partial \upsilon}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right], & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau), & \text{on } \Gamma_1 \times (0, +\infty), \end{cases} \tag{4}$$

and obtained several results concerning global existence and exponential decay rates for various signs of μ_1 and μ_2. The same authors in [7] have studied the following system:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t - \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\left[\dfrac{\partial u}{\partial \upsilon}(x,t) + \delta \dfrac{\partial u_t}{\partial \upsilon}(x,t) - \displaystyle\int_0^t g(t-s)\Delta u(s)\dfrac{\partial u}{\partial \upsilon}(x,s)ds + h(u_t)\right], & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau(t)), & \text{on } \Gamma_1 \times (0, +\infty), \end{cases} \tag{5}$$

with very particular restriction on $h(u_t)$ where they have applied the Faedo–Galerkin method combined with the fixed point theorem, they showed the existence and uniqueness of a local in time solution and under some restrictions on the initial data, the solution continues to exist globally in time. On the other hand, if the interior source dominates the boundary damping, they proved that the solution is unbounded and grows as an exponential function. In addition, in the absence of the strong damping, they proved also the solution ceases to exist and blows up in finite time. Related problem as [6], M. M. Cavalcanti, A. Khemmoudj and M. Medjden [19] studied the following system:

$$\begin{cases} u_{tt} + Au + a(x)g_1(u_t) = 0, & \text{in } \Omega \times (0, +\infty), \\ u_{tt} + \dfrac{\partial u}{\partial \upsilon_A} + A_T \upsilon + g_2(\upsilon_t), & \text{on } \Gamma_1 \times (0, +\infty), \\ u = \upsilon, & \text{on } \Gamma \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ (u(0), \upsilon(0)) = (u^0, \upsilon^0), & \text{in } \Omega \times \Gamma. \\ (u_t(x,0), \upsilon_t(x,0)) = (u_1, \upsilon^1), & \text{in } \Omega \times \Gamma, \end{cases} \tag{6}$$