Computers and Mathematics with Applications 66 (2013) 1245–1260

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator

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ARTICLE INFO

Article history:
Received 22 February 2013
Received in revised form 27 June 2013
Accepted 21 July 2013

Keywords:
Nonlocal diffusion
Nonlocal operators
Nonlocal vector calculus
Fractional Sobolev spaces
Fractional Laplacian
Finite element methods

ABSTRACT

We analyze a nonlocal diffusion operator having as special cases the fractional Laplacian and fractional differential operators that arise in several applications. In our analysis, a nonlocal vector calculus is exploited to define a weak formulation of the nonlocal problem. We demonstrate that, when sufficient conditions on certain kernel functions hold, the solution of the nonlocal equation converges to the solution of the fractional Laplacian equation on bounded domains as the nonlocal interactions become infinite. We also introduce a continuous Galerkin finite element discretization of the nonlocal weak formulation and we derive a priori error estimates. Through several numerical examples we illustrate the theoretical results and we show that by solving the nonlocal problem it is possible to obtain accurate approximations of the solutions of fractional differential equations circumventing the problem of treating infinite-volume constraints.

1. Introduction and motivation

Nonlocal models have been recently used in many applications, including continuum mechanics [1], graph theory [2], nonlocal wave equations [3], and jump processes [4–6]; we consider nonlocal diffusion operators which arise in several and diverse applications such as image analyses [7–10], machine learning [11], nonlocal Dirichlet forms [12], kinetic equations [13,14], phase transitions [15,16], nonlocal heat conduction [17], and the peridynamic model for mechanics [18,1]. In this work we consider a nonlocal integral operator for *anomalous diffusion* which has, as special cases, the fractional Laplacian and fractional derivative operators that are commonly used to model anomalous diffusion [19]. Physical phenomena exhibiting this property cannot be modeled accurately by the usual advection–dispersion equation; among others, we mention turbulent flows [20,21] and chaotic dynamics of classical conservative systems [22].

Nonlocal models differ from the classical partial differential equation models in the fact that in the latter case interactions between two domains occur only due to contact, whereas in the former case interactions can occur at a distance. In particular, let $\Omega \subset \mathbb{R}^n$ denote a bounded, open domain. For $u(\mathbf{x})\colon \Omega \to \mathbb{R}$, define the action of the nonlocal diffusion operator $\mathcal{L}$ on the function $u(\mathbf{x})$ as

$$\mathcal{L}u(\mathbf{x}) := 2\int_{\mathbb{R}^n} \big(u(\mathbf{y}) - u(\mathbf{x})\big)\,\gamma(\mathbf{x},\mathbf{y})\,d\mathbf{y} \quad \forall\,\mathbf{x} \in \Omega \subseteq \mathbb{R}^n,$$

where the volume of Ω is non-zero and the *kernel* $\gamma(\mathbf{x},\mathbf{y})\colon \Omega \times \Omega \to \mathbb{R}$ is a non-negative symmetric mapping. We are interested in the nonlocal, steady-state diffusion equation

$$\begin{cases} -\mathcal{L}u = f & \text{on } \Omega \\ u = 0 & \text{on } \Omega_{\mathcal{I}}, \end{cases}$$

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0898-1221/$ – see front matter
http://dx.doi.org/10.1016/j.camwa.2013.07.022

where the equality constraint (extension of a Dirichlet boundary condition for differential problems) acts on an interaction volume $\Omega_{\mathcal{I}}$ that is disjoint from Ω.

The numerical solution of fractional differential equations is an open problem and it is the object of current research in applications of models of fractional order; see [23] for recent work including many citations to the literature. Common techniques include methods that take advantage of Laplace and Fourier transforms to obtain classical solutions [24,25] and finite difference methods [26–30] used for constructing numerical approximations. Galerkin discretizations and their error analysis have been considered in [31,32] for the discretization of the steady state fractional advection–diffusion equation.

A goal of this paper is to develop and analyze discretization methods for fractional Laplacian equations on bounded domains. We do this by exploiting the fact that *the fractional Laplacian operator* $(-\Delta)^s$ *is a special case of the nonlocal operator* $\mathcal{L}$. In particular, we compare the solution of the nonlocal steady diffusion equation with the solution of the fractional Laplacian equation on bounded domains and show that solving nonlocal problems is a viable alternative to solving fractional differential equations. A main contribution of this work is to show that not only $(-\Delta)^s$ is a special case of $\mathcal{L}$, but that it is the limit of the nonlocal operator as the nonlocal interactions become infinite, provided that sufficient conditions on certain kernel functions hold. This fact has important consequences in the treatment of problems involving the fractional Laplacian equations on bounded domains. In fact, nonlocal problems are a well posed and are a more general formulation of fractional differential models; this is useful for both the analysis of fractional differential equations and for developing finite-dimensional discretization schemes. In [33], finite-dimensional approximations of nonlocal problems are discussed, including finite element discretizations; Galerkin formulations are introduced and results are proved about their well posedness and about estimates for the approximation error and the condition number of finite element matrices. This helps in designing efficient numerical methods for the solution of nonlocal diffusion problems and, as a consequence, of fractional differential problems. Also, being able to quantify the discrepancy between nonlocal solutions and solutions of fractional differential equations (as we demonstrate in this paper) allows us to determine to what extent the former are accurate approximations of the latter. Furthermore, an important advantage of approximating the solution of fractional differential problems with nonlocal solutions is that in the latter case we do not have to deal with the infinite-volume constraints as happens for the solution of fractional differential equations on bounded domains where some expedients for constructing finite-dimensional approximations have to be introduced.

We note that the analysis and approximation of fractional Laplacian problems for the case $\Omega = \mathbb{R}^n$, though of interest in many applications, is not a goal of this work and is not discussed. Here, our interest is strictly on the bounded domain problems.

We also note that a significant advantage of recasting fractional Laplacian problems in terms of the nonlocal problems we consider is that, for the first time, such problems can be treated on bounded domains for the case of $s \leq 1/2$. Indeed, the key to this is our introduction of volume constraints as a generalization of boundary conditions; the latter is not well defined for $s \leq 1/2$ because traces of functions in the energy space associated with fractional Laplacian operators are themselves not well defined.

In our analysis, a recently developed nonlocal vector calculus [34] is exploited to define a weak formulation of the nonlocal problem and to prove the convergence of the nonlocal solution to the solution of the fractional differential problem. In Section 2, we provide a brief review of those aspects of the nonlocal calculus that are useful in the remainder of the paper, introduce the kernel function, and discuss its properties. In Section 3, we define the fractional Laplacian as a special case of the nonlocal operator $\mathcal{L}$ and prove the convergence of the nonlocal operator to the fractional Laplacian as the nonlocal interactions become infinite. In Section 4 we introduce finite-dimensional discretizations of the nonlocal problem and we study the convergence of the approximate nonlocal solutions to the solution of the fractional Laplacian equation. We also discuss the choice of nonuniform grids for mitigating the high computational costs that occur when the extent of nonlocal interactions becomes large. In Section 5, we present the results of some numerical tests for finite element discretizations of one-dimensional problems; by providing qualitative and quantitative comparisons between approximate nonlocal solutions and exact solutions of the fractional Laplacian equation, these results illustrate the theory introduced in Sections 3 and 4.

2. Elements of a nonlocal vector calculus

In this section, we review relevant aspects of the nonlocal calculus including nonlocal operators, kernel functions, and nonlocal function spaces. Details about the nonlocal calculus are found in [34].

The action of the *nonlocal divergence operator* $\mathcal{D}\colon \mathbb{R}^n \to \mathbb{R}$ on $\boldsymbol{\nu}$ is defined as

$$\mathcal{D}\big(\boldsymbol{\nu}\big)(\mathbf{x}) := \int_{\mathbb{R}^n} \big(\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) + \boldsymbol{\nu}(\mathbf{y},\mathbf{x})\big)\cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad \text{for } \mathbf{x}\in\mathbb{R}^n, \tag{1a}$$

where $\boldsymbol{\nu}(\mathbf{x},\mathbf{y}), \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ with $\boldsymbol{\alpha}$ antisymmetric, i.e., $\boldsymbol{\alpha}(\mathbf{y},\mathbf{x}) = -\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})$, are given mappings. This definition is justified in [34]. The action of the *nonlocal gradient operator* $\mathcal{G}\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ on u is defined as

$$\mathcal{G}\big(u\big)(\mathbf{x},\mathbf{y}) := \big(u(\mathbf{y}) - u(\mathbf{x})\big)\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \quad \text{for } \mathbf{x},\mathbf{y}\in\mathbb{R}^n, \tag{1b}$$

where $u(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}$ is a given mapping. The fact that $-\mathcal{G}$ and $\mathcal{D}$ are adjoint operators is shown in [34]; in fact, in [34], the operator $\mathcal{G}$ is denoted by $-\mathcal{D}^*$.