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New approach to study splines by blossoming method and application to the construction of a bivariate C^1 quartic quasi-interpolant☆

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ABSTRACT

This work is a contribution in the approximation theory for studying and analyzing piecewise polynomial functions (splines), which uses the blossoming approach. Some existing results in the literature are reformulated, such as the smoothness conditions between polynomials of a spline, by using the affinity property of the blossom. Some definitions of sub-splines are proposed which can be very useful in the study and the construction of splines such as macro-elements or quasi-interpolants. As an application of the proposed results, a C^1 quartic spline quasi-interpolant with optimal approximation order is defined without using any mask for smoothness or B-spline basis. Numerical results are presented and compared with other methods given in the literature.

1. Introduction

The blossoms of polynomials or of splines are a good tool for analyzing and studying piecewise polynomial functions. This approach, introduced by de Casteljau [1], Ramshaw [2,3] and others (see [4–8] and references therein), provides an elegant and powerful tool to construct spline curves or surfaces. In addition to their benefits in CAGD, they present other advantages in the approximation field. For example, the coefficients of a polynomial in Bernstein basis (respectively in B-spline basis) are obtained as the blossoms of the considered polynomial at some specific points. This representation, named Marsden identity, seems to be very interesting for the construction, in a simple way, of different types of quasi-interpolants see [9–19] and references therein.

In the first part of this work, we propose to shed more light on the connection between blossoms and splines. Although some results are well known in the literature (see [20], for instance), we give other versions of these results in a simple and useful form, especially those related to the smoothness conditions between polynomial pieces of a regular spline S. We also give some definitions of sub-splines of S written in terms of the blossom of S. These results are very useful for the construction and the study of smooth piecewise polynomial functions as for instance the construction of macro-elements or quasi-interpolants.

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Usually a discrete quasi-interpolant for a given function f, noted $\mathcal{Q}f$, is obtained as linear combination of some elements of a suitable set of basis functions. In order to achieve local control, these functions are required to be positive, to ensure stability and to have small local supports. Other choice of the basis function can be given by solving a minimization problem (see [21], for instance). The coefficients of the linear combination are the values of linear functionals depending on f. The exactness of the quasi-interpolation operator $\mathcal{Q}$ on the appropriate space of polynomials is achieved using well known results. Various methods for building bivariate discrete quasi-interpolants have been developed in the literature (see for examples [22–29] and references therein).

Recently, another idea has been introduced by T. Sorokina and Zeilfelder in [30] to define a C^1 quartic quasi-interpolation $\mathcal{Q}f$ on a uniform three-directional mesh to approximate regularly distributed data. In this method, the B-coefficients of the restriction of the quasi-interpolant to each triangle in the partition are set in such a way that the C^1 smoothness is automatically satisfied. This construction is similar to the approach for C^1 quadratic given in [31]. The B-coefficients of $\mathcal{Q}f$ are immediately available from the given values by applying local averaging and the coefficient masks. Despite the elegance of this method, the obtained quasi-interpolant is constructed only on a particular triangulation and does not reproduce the whole space $\mathbb{P}_4(\mathbb{R}^2)$ of bivariate polynomials of degree less than or equal to 4, it is only fourth order accurate. For this reason, additional work is required in order to define a quasi-interpolation scheme exact on $\mathbb{P}_4(\mathbb{R}^2)$.

In the second part of our work, we apply the results developed in the first part to propose a method, which differs from the two preceding ones, to construct a spline quasi-interpolant by a two-stage algorithm. In particular, we describe a new method to construct on arbitrary triangulation a C^1 quartic quasi-interpolant spline with optimal approximation order without using any coefficient mask for smoothness like in [30] or quartic B-splines basis [32–36]. Most of the B-coefficients of $\mathcal{Q}f$ are obtained on the first stage of the proposed algorithm by computing the blossoms of some local polynomials, so that the most of the smoothness conditions between the piecewise polynomials of $\mathcal{Q}f$ are automatically satisfied. The remaining B-coefficients around a fixed vertex are obtained by imposing only one smoothness condition across the edges of the molecule centered at this vertex. The constructed quasi-interpolant depends on some parameters which can be well selected in order to improve the approximation error.

The paper is organized as follows. In Section 2, we give some definitions and properties of bivariate polynomials and their polar forms. In Section 3, we reformulate some existing results in the literature, such as the smoothness conditions between polynomials of a spline and we propose some definitions of sub-splines of a given spline. In Section 4, we describe a method to construct a parameterized spline quasi-interpolant of degree 4 and class 1 on arbitrary triangulation with optimal approximation order. Finally, in Section 5, we discuss good choices of the quasi-interpolation parameters and we propose some numerical examples to illustrate the theoretical results.

2. Preliminaries

2.1. Polynomials on triangles

Consider a non-degenerated triangle $\mathcal{T}(V_1, V_2, V_3)$ in the plane, having vertices V_i with Cartesian coordinates $(x_i, y_i),\ i = 1, 2, 3$. The barycentric coordinates $\lambda_{\mathcal{T}} := (\lambda_{\mathcal{T},1}, \lambda_{\mathcal{T},2}, \lambda_{\mathcal{T},3})$ of an arbitrary point $(x, y) \in \mathbb{R}^2$ with respect to $\mathcal{T}$ are defined as the unique solution of the system

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{\mathcal{T},1} \\ \lambda_{\mathcal{T},2} \\ \lambda_{\mathcal{T},3} \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{2.1}$$

We denote by $\mathbb{P}_k(\mathbb{R}^2)$ the linear space of bivariate polynomials of degree less than or equal to k. Set

$$\mathcal{I}_k := \{(i_1, i_2, i_3) \in \mathbb{N}^3,\ |i| = i_1 + i_2 + i_3 = k\}.$$

Then any polynomial $P \in \mathbb{P}_k(\mathbb{R}^2)$ on $\mathcal{T}$ has a unique representation

$$P(x, y) := b(\lambda_{\mathcal{T}}) = \sum_{i \in \mathcal{I}_k} b_{i,\mathcal{T}} B^{(k)}_{i,\mathcal{T}}(\lambda_{\mathcal{T}}), \tag{2.2}$$

where

$$B^{(k)}_{i,\mathcal{T}}(\lambda_{\mathcal{T}}) = \frac{m!}{i_1! i_2! i_3!} \lambda^{i_1}_{\mathcal{T},1} \lambda^{i_2}_{\mathcal{T},2} \lambda^{i_3}_{\mathcal{T},3} \tag{2.3}$$

are the Bernstein–Bézier polynomials of degree k. The coefficients $b_{i,\mathcal{T}}$ are called the Bézier ordinates of the polynomial P with respect to the triangle $\mathcal{T}$.

Let Ω be a simply connected subset of $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$ and let Δ be a triangulation of Ω having vertices V_i with Cartesian coordinates (x_i, y_i), $i = 1, \ldots, N_v$. For each vertex V_i of Δ, we denote by $\mathcal{M}_{V_i}$ the set of all triangles of