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A priori and a posteriori error analyses of a pseudostress-based mixed formulation for linear elasticity[☆]

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ABSTRACT

In this paper we present the a priori and a posteriori error analyses of a non-standard mixed finite element method for the linear elasticity problem with non-homogeneous Dirichlet boundary conditions. More precisely, the approach introduced here is based on a simplified interpretation of the pseudostress–displacement formulation originally proposed in Arnold and Falk (1988), which does not require symmetric tensor spaces in the finite element discretization. In addition, physical quantities such as the stress, the strain tensor of small deformations, and the rotation, are computed through a simple postprocessing in terms of the pseudostress variable. Furthermore, we also introduce a second element-by-element postprocessing formula for the stress, which yields an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm. We apply the classical Babuška–Brezzi theory to prove that the corresponding continuous and discrete schemes are well-posed. In particular, Raviart–Thomas spaces of order $k \geq 0$ for the pseudostress and piecewise polynomials of degree $\leq k$ for the displacement can be utilized. Moreover, we remark that in the 3D case the number of unknowns behaves approximately as 9 times the number of elements (tetrahedra) of the triangulation when $k = 0$. This factor increases to 12.5 when one uses the classical PEERS. Next, we derive a reliable and efficient residual-based a posteriori error estimator for the mixed finite element scheme. Finally, several numerical results illustrating the performance of the method, confirming the theoretical properties of the estimator, and showing the expected behaviour of the associated adaptive algorithm, are provided.

1. Introduction

The introduction of further unknowns of physical interest, such as stresses, rotations, and tractions, and the need of locking-free numerical schemes when the corresponding Poisson ratio approaches 1/2, have historically been the main

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reasons for the utilization of dual-mixed variational formulations and their associated mixed finite element methods to solve elasticity problems. The incompressible case can also be easily handled with this kind of formulations since the constants appearing in the stability and a priori error estimates do not depend on the unbounded Lamé parameter. Consequently, the derivation of appropriate finite element subspaces yielding corresponding well-posed Galerkin schemes has been extensively studied in the last three decades at least, and important early contributions with weakly imposed symmetry for the stress, which include the classical PEERS element and related approaches, were provided in [1–3], to name a few. However, since the appearing of those first works, the main challenge in this direction has been the development of mixed finite element methods that incorporate the symmetry of the stress into the definition of the respective continuous and discrete spaces. In fact, the first stable mixed finite element methods for linear elasticity, with symmetric and weakly symmetric stresses, were derived, thanks to the finite element exterior calculus, about a decade ago (see, e.g. [4–7]). In particular, the polynomial shape-functions provided in [7] yield the first stable elements for the symmetric stress–displacement approach in two dimensions. In this case, 24 degrees of freedom defining piecewise cubic polynomials for the stresses, and piecewise linear functions for the displacement, constitute the cheapest element. In turn, 162 degrees of freedom defining piecewise quartic stresses, and again piecewise linear displacements, form the lowest order element in 3D, which was introduced in [8]. Furthermore, stable elements with weak imposition of symmetric stresses have been derived in [4,6]. The corresponding element using the lowest polynomial degrees is determined by piecewise constants approximations for both the displacement and rotation, and piecewise linear functions for the stress. In addition, stable Stokes elements and interpolation operators keeping the reduced symmetry were employed in [9] to derive simpler proofs of the main results provided in [4,6].

On the other hand, an alternative way of dealing with dual-mixed variational formulations in continuum mechanics, without the need of imposing neither strong nor weak symmetry of the stresses, is given by the utilization of pseudostress-based approaches. In fact, this technique, which has become very popular, specially in fluid mechanics, has gained considerable attention in recent years due to its applicability to diverse linear as well as nonlinear problems. In particular, the velocity–pseudostress formulation of the Stokes equations was first studied in [10], and then reconsidered in [11], where further results, including the eventual incorporation of the pressure unknown and an associated a posteriori error analysis, were provided. In turn, augmented mixed finite element methods for pseudostress-based formulations of the stationary Stokes equations, which extend analogue results for linear elasticity problems (see [12–14]), were introduced and analysed in [15]. Furthermore, the velocity–pressure–pseudostress formulation has also been applied to nonlinear Stokes problems. In particular, a new mixed finite element method for a class of models arising in quasi-Newtonian fluids, was introduced in [16]. The results in [16] were extended in [17] to a setting in reflexive Banach spaces, thus allowing other nonlinear models such as the Carreau law for viscoplastic flows. Moreover, the dual-mixed approach from [16,17] was reformulated in [18] by restricting the space for the velocity gradient to that of trace-free tensors. For related contributions dealing with pseudostress-based formulations in incompressible flows, we refer to [19,20], and the references therein. In turn, the corresponding extension to the Navier–Stokes equations has been developed in [21,22]. More recently, a new dual-mixed method for the aforementioned problem, in which the main unknowns are given by the velocity, its gradient, and a modified nonlinear pseudostress tensor linking the usual stress and the convective term, has been proposed in [23]. The idea from [23] has been modified in [24] through the introduction of a nonlinear pseudostress tensor linking now the pseudostress (instead of the stress) and the convective term, which, together with the velocity, constitute the only unknowns. Lately, the approach from [24] has been further extended in [25,26], where new augmented mixed-primal formulations for the stationary Boussinesq problem and the Navier–Stokes equations with variable viscosity, respectively, have been proposed and analysed. In spite of the many aforedescribed works, it is quite surprising to realize that almost no contribution is available in the literature on the use of pseudostress-based formulations for the elasticity problem. Indeed, the search in MathScinet under the title words ''*pseudostress*'' and ''*elasticity*'' yields no results at all. Actually, up to the authors' knowledge, the only paper referring to this issue is [27], where a modified Hellinger–Reissner principle is employed to derive a new mixed variational formulation for the equations of linear elasticity. The resulting approach yields a pseudostress unknown defined in terms of the gradient of the displacement field, but depending also on a parameter to be chosen conveniently.

In the present paper we modify the approach from [27] by realizing that, under a suitable rewriting of the equilibrium equation, one can define a simpler pseudostress unknown in terms again of the gradient of the displacement field, but independent of any additional parameter. In addition, we introduce an element-by-element postprocessing formula for the symmetric stress, which yields an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm. Moreover, a reliable and efficient residual-based a posteriori error estimator for the mixed finite element scheme is also derived. A very summarized version of the first part of this work is available in [28]. The rest of this paper is organized as follows. In Section 2 we describe the linear elasticity problem with non-homogeneous Dirichlet boundary conditions, derive its pseudostress-based dual-mixed formulation, and then show that it is well-posed. In Section 3 we introduce and analyse the associated mixed finite element method. In particular, we show that Raviart–Thomas spaces of order $k \geq 0$ for the pseudostress and piecewise polynomials of degree $\leq k$ for the displacement can be employed, which, in the 3D case, yields a global number of unknowns behaving approximately as only 9 times the number of tetrahedra of the triangulation when $k = 0$. Next, a reliable and efficient residual-based a posteriori error estimator is developed in Section 4. Finally, several numerical results showing the good performance of the mixed finite element method, confirming the reliability and efficiency of the estimator, and illustrating the expected behaviour of the associated adaptive algorithm, are reported in Section 5.