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Ground states for diffusion system with periodic and asymptotically periodic nonlinearity☆

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ABSTRACT

In this paper, we study the following diffusion system

$$\begin{cases} \partial_t u - \Delta_x u + b(t,x)\cdot\nabla_x u + V(x)u = g(t,x,v), \\ -\partial_t v - \Delta_x v - b(t,x)\cdot\nabla_x v + V(x)v = f(t,x,u) \end{cases}$$

where $b \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ and $V(x) \in C(\mathbb{R}^N, \mathbb{R})$. By using non-Nehari manifold method developed in Tang (2014), we investigate the existence of the ground state solutions of Nehari type for the above problem with periodic and asymptotically periodic nonlinearity.

1. Introduction and main results

We study the following diffusion system on $\mathbb{R}\times\mathbb{R}^N$

$$\begin{cases} \partial_t u - \Delta_x u + b(t,x)\cdot\nabla_x u + V(x)u = g(t,x,v), \\ -\partial_t v - \Delta_x v - b(t,x)\cdot\nabla_x v + V(x)v = f(t,x,u) \end{cases} \tag{1.1}$$

where $z := (u, v) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^2, b = (b_1,\ldots,b_N) \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ with the gauge condition $\operatorname{div} b(t,x) = 0$ ($\operatorname{div} b(t,x) := \sum_{i=1}^N \partial_{x_i} b_i(t,x)$), $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $g(t,x,v), f(t,x,u) \in C(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}, \mathbb{R})$. Such problem arises in control of systems governed by partial differential equations and is related to the Schrödinger equations (see [1,2]). In this paper, we are interested in the existence of ground state solutions of Nehari type of system (1.1).

For the case of a bounded domain the systems like or similar to (1.1) were studied by a number of authors. For instance, see [3–11] and the references therein. When assuming $b(t,x) = 0$, $V(x) = 0$, Brézis and Nirenberg [3] considered the following system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \quad \text{in } (0,T)\times\Omega.$$

Using Schauder's fixed point theorem, they obtained a solution (u, v) with $u \in L^4$ and $v \in L^6$. In [4], Clément et al. considered the problem

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2}v \\ -\partial_t v - \Delta_x v = |u|^{p-2}u \end{cases} \quad \text{in } (-T,T)\times\Omega,$$

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where p, q satisfy

$$\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1.$$

The existence of positive periodic solution was obtained by using a mountain pass argument. Recently, based on a local linking theorem, Mao et al. [10,11] proved the existence of periodic solution. For other related elliptic system problems, we refer readers to [5,6,8,9].

The problem in the whole space $\mathbb{R}^N$ was considered recently in some works. Assuming $b(t,x) = 0$, $V(x) \neq 0$, Bartsch and Ding [12] dealt with the problem under the classic Ambrosetti–Rabinowitz condition

$$0 < \mu H(t,x,z) \le H_z(t,x,z)z, \quad z \neq 0 \tag{1.2}$$

for $\mu > 2$ and

$$|H_z(t,x,z)|^{\nu} \le c H_z(t,x,z)z, \quad \text{for } |z| > 1 \tag{1.3}$$

for some $1 + N/(N+4) < \nu < 2$. Assumptions (1.2) and (1.3) were improved later by Schechter and Zou in [13]. Recently, Ding et al. [14] and Wei and Yang [15] considered the case $b(t,x) \neq 0$ via variational methods. Under periodic assumption, the existence of infinitely many solutions was obtained for both superquadratic and asymptotically quadratic cases when the nonlinearity is symmetric. Without the symmetric assumption, Wang et al. [16] also obtained infinitely many solutions by using a reduction method. For non-periodic case, we refer readers to [17–21] and the references therein.

Motivated by these works, in the present paper, we consider the periodic and asymptotically periodic cases with more generic conditions, and prove the existence of ground state solutions. For convenience of proceeding further, we need to make the following assumptions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i for $i = 1, \dots, N$ and $a := \min_{x\in\mathbb{R}^N} V(x) > 0$;
(B) $b \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ is 1-periodic in t and x_i for $i = 1, \dots, N$ and $\operatorname{div} b = 0$;
(F_1) $f(t,x,s)$ and $g(t,x,s)$ are continuous and there is a constant $C > 0$ such that

$$|f(t,x,s)| \le C(1+|s|^{p-1}) \quad \text{and} \quad |g(t,x,s)| \le C(1+|s|^{p-1}) \quad \text{for all } (t,x,s),$$

where $p \in (2, N^*)$, $N^* = \infty$ if $N = 1$ and $N^* = \frac{2(N+2)}{N}$ if $N \ge 2$;
(F_2) $f(t,x,s) = o(|s|)$ and $g(t,x,s) = o(|s|)$ as $|s| \to 0$ uniformly in (t,x);
(F_3) $\lim_{|s|\to\infty} \frac{F(t,x,s)}{|s|^2} = \infty$ and $\lim_{|s|\to\infty} \frac{G(t,x,s)}{|s|^2} = \infty$ uniformly in (t,x);
(F_4) $s \mapsto \frac{f(t,x,s)}{|s|}$ and $s \mapsto \frac{g(t,x,s)}{|s|}$ are nondecreasing on $(-\infty,0)$ and $(0,+\infty)$;
(F_5) $f(t,x,s)$ and $g(t,x,s)$ are 1-periodic in t and x_i for $i = 1, \dots, N$;
(F_5') $f(t,x,s) = f_0(t,x,s) + f_1(t,x,s)$, $g(t,x,s) = g_0(t,x,s) + g_1(t,x,s)$, $f_0, g_0 \in C(\mathbb{R}^N \times \mathbb{R})$, $f_0(x,t)$ and $g_0(t,x,s)$ are 1-periodic in t and x_i for $i = 1, \dots, N$; $\frac{f_0(t,x,s)}{|s|}$ and $\frac{g_0(t,x,s)}{|s|}$ are nondecreasing on $(-\infty,0)$ and $(0,+\infty)$, and there exists a function $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} a(x) = 0$ such that

$$0 < sf_1(t,x,s) \le a(x)\left(|s|^2 + |s|^p\right), \qquad 0 < sg_1(t,x,s) \le a(x)\left(|s|^2 + |s|^p\right).$$

It is well known that the Nehari manifold method is a effective tool for studying the existence of ground state solution, see [22,23] for positive definite Schrödinger equation. However, for the indefinite variational problems, the usual Nehari manifold method cannot be applied directly since the set $\mathcal{N} := \{u \in E : u \neq 0 \text{ and } \langle \Phi'(u), u\rangle = 0\}$ need not be closed (see [24] Chap. 3 for a description and some applications of this method). Recently, Szulkin and Weth [25] developed a powerful approach to treat the strongly indefinite periodic Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

More precisely, they used the generalized Nehari manifold (which was first introduced in [26] for smooth case) to construct a natural constrained problem and obtained the ground state solution under some standard assumptions and the following Nehari type monotone condition

(N_0) $\frac{f(x,s)}{|s|}$ is strictly increasing in s on $\mathbb{R}\backslash\{0\}$ for every $x \in \mathbb{R}^N$.

As we know, condition (N_0) plays a very important role in generalized Nehari manifold method. In the present paper, the Nehari type monotone condition is no longer holds, hence their arguments collapses. In order to successfully carry out our work, we use the non-Nehari manifold method developed by Tang [27] to establish the existence of ground state solution of Nehari type in two cases: the periodic case and the asymptotically periodic case. The main idea of this approach is to find a minimizing Cerami sequence for energy functional Φ outside $\mathcal{M}$ by using the diagonal method, where

$$\mathcal{M} := \{z \in E\backslash E^- : \langle \Phi'(z), z\rangle = \langle \Phi'(z), \varphi\rangle = 0, \ \forall \varphi \in E^-\}$$

and Φ, E, E^- will be defined in Section 2. Additionally, based on the linking theorem in [28,29], there are also many works devoted to the existence of least energy solution for periodic Schrödinger equation, elliptic system and Hamiltonian system. For example, see [30–45] and the references therein.

We are now in a position to state the main results of this paper.