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Hamiltonian-connectivity and strongly Hamiltonian-laceability of folded hypercubes☆

Sun-Yuan Hsieh*, Che-Nan Kuo

Department of Computer Science and Information Engineering, National Cheng Kung University, No. 1, University Road, Tainan 70101, Taiwan

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Abstract

In this paper, we analyze a hypercube-like structure, called the folded hypercube, which is basically a standard hypercube with some extra links established between its nodes. We first show that the n-dimensional folded hypercube is bipartite when n is odd. We also show that the n-dimensional folded hypercube is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or n (≥ 2) is even.

Keywords: Hypercubes; Folded hypercubes; Hamiltonian-connectivity; Strongly Hamiltonian-laceability; Bipartite graphs

1. Introduction

Design of interconnection networks is an important integral part of parallel processing and distributed systems. There are a large number of topological choices for interconnection networks (the interested readers may refer to [1–3] for extensive references). Among them, the hypercube [4] has several excellent properties such as recursive structure, regularity, symmetry, small diameter, relatively short mean internode distance, low degree, and very small link complexity, which are very important for designing massively parallel or distributed systems [5]. Since its introduction, many variants of the hypercube have been proposed [6–8]. One variant that has been the focus of a great deal of research is the *folded hypercube*, which is an extension of the hypercube, constructed by adding a link to every pair of nodes that are the farthest apart, i.e., two nodes with complementary addresses. The folded hypercube has been shown to be able to improve the system's performance over a regular hypercube in many measurements [6,9].

$G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. We also use $V(G)$ and $E(G)$ to denote the node set and edge set of G, respectively. In this paper, we use graph and network, node and vertex, link and edge, interchangeably. Usually when the Hamiltonicity of a graph G is concerned, it is investigated whether G is Hamiltonian or Hamiltonian-connected. A cycle (resp., path) in G is called a *Hamiltonian cycle* (resp., *Hamiltonian path*) if it contains every node of G exactly

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* Corresponding author. Tel.: +886 6 2757575x62538; fax: +886 6 274 7076.
E-mail address: hsiehsy@mail.ncku.edu.tw (S.-Y. Hsieh).

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once. G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if there exists a Hamiltonian path between every two nodes of G.

A graph $G = (V_0 \bigcup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y)|x \in V_0 \text{ and } y \in V_1\}$. We call V_0 and V_1 *partite sets* of G, and $V_0 \bigcup V_1$ a *bipartition*. Hypercubes [5] and star graphs [10] are both bipartite. However, since a bipartite graph is not Hamiltonian-connected except for K_2 or K_1, Simmons [11] introduced the concept of Hamiltonian-laceability for those Hamiltonian bipartite graphs. A Hamiltonian bipartite graph $G = (V_0 \bigcup V_1, E)$ is *Hamiltonian-laceable* if there is a Hamiltonian path between any two nodes x and y, where $x \in V_0$ and $y \in V_1$. Hsieh et al. [12] further extended this concept and proposed the concept of *strongly Hamiltonian-laceability*. A Hamiltonian-laceable graph $G = (V_0 \bigcup V_1, E)$ is *strongly* Hamiltonian-laceable if there is a simple path of length $|V_0| + |V_1| - 2$ between any two nodes of the same partite set. Simmons [11] showed that the n-dimensional hypercube is Hamiltonian-laceable. Tsai et al. [13] further showed that the n-dimensional hypercube is strongly Hamiltonian-laceable. Hsieh et al. [12] showed that the n-dimensional star graph is strongly Hamiltonian-laceable. In this paper, we first show that the n-dimensional folded hypercube is bipartite when n is odd. We also show that the n-dimensional folded hypercube is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or n (≥ 2) is even.

2. Preliminaries

An n-dimensional hypercube (n-cube for short) can be represented as an undirected graph $Q_n = (V, E)$ such that V consists of 2^n nodes which are labeled as binary numbers of length n from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$. E is the set of edges that connects two nodes if and only if they differ in exactly one bit of their labels. Thus, each node has immediate links with exactly n other nodes. It can easily be shown that $|E| = n2^{n-1}$. A link (or edge) $e = (v_i, v_j) \in E$ represents the two nodes v_i and v_j which are linked by e and have exactly one bit different. Therefore e can be denoted using the two nodes that it links: If $v_i = b_n b_{n-1} \ldots b_k \ldots b_1$, $v_j = b_n b_{n-1} \ldots \bar{b}_k \ldots b_1$ (where $b_l \in \{0, 1\}, l = 1, \ldots, n$), and $e = (v_i, v_j)$, then we denote e as $b_n \ldots b_{k+1} x b_{k-1} \ldots b_1$. We call $b_n \ldots b_{k+1} x b_{k-1} \ldots b_1$ a link of dimension k. There are 2^{n-1} links in each dimension.

Let $x = x_n x_{n-1} \ldots x_1$ be an n-bit binary string. For $1 \leq k \leq n$, we use $x^{(k)}$ to denote the binary strings $y_n y_{n-1} \ldots y_1$ such that $y_k = 1 - x_k$ and $x_i = y_i$ for all $i \neq k$. The *Hamming weight* $hw(x)$ of x is the number of i's such that $x_i = 1$. Let $x = x_n x_{n-1} \ldots x_1$ and $y = y_n y_{n-1} \ldots y_1$ be two n-bit binary strings. The *Hamming distance* $h(x, y)$ between two nodes x and y is the number of different bits in the corresponding strings of both nodes. Note that an n-cube Q_n is a bipartite graph with bipartition $\{x|hw(x) \text{ is odd}\}$ and $\{x|hw(x) \text{ is even}\}$. Let $d_{Q_n}(x, y)$ be the distance of a shortest path between two vertices x and y in graph Q_n. Then, it is known that $d_{Q_n}(x, y) = h(x, y)$.

An *n-dimensional folded hypercube* (*folded n-cube* for short) FQ_n is a regular n-dimensional hypercube augmented by adding more links among its nodes. More specifically, a folded n-cube is obtained by adding a link between two nodes whose addresses are complementary to each other; i.e., for a node whose address is $b = b_1 b_2 \ldots b_n$, it now has one more link to node $\bar{b} = \bar{b}_1 \bar{b}_2 \ldots \bar{b}_n$, in addition to its original n links. So a folded n-cube has 2^{n-1} more links than a regular n-cube. We call these augmented links *skips*, to distinguish them from regular links, and use S to denote the set of skips. So the complete link set $E(FQ_n)$ of a folded hypercube can be expressed as $I \cup S$. In other words, we can formally define the edges of a folded n-cube by $E(FQ_n) = I \cup S = \{e = (u, v)|h(u, v) = 1 \in I \text{ or } h(u, v) = n \in S\}$. Fig. 1 illustrates a two-dimensional and a three-dimensional folded hypercube.

For convenience, a folded n-cube FQ_n can be represented with $\underbrace{** \ldots **}_{n} = *^n$, where $* \in \{0, 1\}$ means the "*don't care*" symbol. Moreover, $Q_{n-1}^{0i} = *^{n-i}\, 0 *^{i-1}$ and $Q_{n-1}^{1i} = *^{n-i}\, 1 *^{i-1}$, which contain the nodes with the ith bits 0 and 1, respectively, represent two node-disjoint $(n-1)$-cubes. Formally, Q_{n-1}^{0i} (resp., Q_{n-1}^{1i}) is the subgraph of FQ_n induced by $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|x_i = 0\}$ (resp., $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|x_i = 1\}$). Clearly, each Q_{n-1}^{ji}, $j \in \{0, 1\}$, is isomorphic to Q_{n-1}.

Definition 1. An *i-partition on* $FQ_n = *^n$, where $1 \leq i \leq n$, is a partition of FQ_n along dimension i into two $(n-1)$-cubes $*^{n-i}\, 0 *^{i-1}$ and $*^{n-i}\, 1 *^{i-1}$.