Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 1119–1128
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Decomposition formulas associated with the Lauricella multivariable hypergeometric functions

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Received 7 July 2006; accepted 17 July 2006

Abstract

By making use of some techniques based upon certain inverse pairs of symbolic operators, the authors investigate several decomposition formulas associated with Lauricella's hypergeometric functions $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ in r variables. In the three-variable case when $r = 3$, some of these operational representations are constructed and applied in order to derive the corresponding decomposition formulas. With the help of these inverse pairs of symbolic operators, a total of 15 decomposition formulas are found, which are expressed as products of hypergeometric functions of the Gauss and Appell types.

Keywords: Lauricella hypergeometric functions; Multiple hypergeometric functions; Inverse pairs of symbolic operators; Generalized hypergeometric function; Integral representations; Confluent hypergeometric functions

1. Introduction and definitions

Multiple hypergeometric functions (that is, hypergeometric functions in several variables) occur naturally in a wide variety of problems (see, for details, [1]). In particular, the Lauricella functions $F_A^{(r)}, \ldots, F_D^{(r)}$ in r (real or complex) variables, defined by ([2] and [1, p. 33])

$$F_A^{(r)}(\alpha, \beta_1, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)$$
$$:= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r}(\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_r)_{m_r}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \quad (|x_1| + \cdots + |x_r| < 1), \tag{1.1}$$

$$F_B^{(r)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r; \gamma; x_1, \ldots, x_r)$$
$$:= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_r)_{m_r}(\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma)_{m_1+\cdots+m_r}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \quad (\max\{|x_1|, \ldots, |x_r|\} < 1), \tag{1.2}$$

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.07.007

$$F_C^{(r)}(\alpha, \beta; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)$$
$$:= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r} (\beta)_{m_1+\cdots+m_r}}{(\gamma_1)_{m_1} \cdots (\gamma_r)_{m_r}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \quad \left(\sqrt{|x_1|} + \cdots + \sqrt{|x_r|} < 1\right), \tag{1.3}$$

and

$$F_D^{(r)}(\alpha, \beta_1, \ldots, \beta_r; \gamma; x_1, \ldots, x_r) := \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r} (\beta_1)_{m_1} \cdots (\beta_r)_{m_r}}{(\gamma)_{m_1+\cdots+m_r}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}$$
$$(\max\{|x_1|, \ldots, |x_r|\} < 1), \tag{1.4}$$

together with their special cases when $r = 2$ (namely, the Appell functions F_2, F_3, F_4, and F_1, respectively) arise frequently in various physical and quantum chemical applications ([1]; see also the recent works [3,4] and the references cited therein).

For various multivariable hypergeometric functions including (for example) the Lauricella multivariable function $F_A^{(r)}$ defined by (1.1), Hasanov and Srivastava [5] presented a number of decomposition formulas in terms of such simpler hypergeometric functions as the Gauss and Appell functions. The main object of this sequel to the work of Hasanov and Srivastava [5] is to show how some rather elementary techniques based upon certain inverse pairs of symbolic operators would lead us easily to several decomposition formulas associated with Lauricella's hypergeometric function $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ in r variables ($r = 2, 3, 4, \ldots$) and with other multiple hypergeometric functions.

Over six decades ago, Burchnall and Chaundy [6,7] (and Chaundy [8]) systematically presented a number of expansion and decomposition formulas for double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(h)_k\, k!} \tag{1.5}$$

and

$$\Delta_{xy}(h) := \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\, k!}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (\delta_1+h)_k (\delta_2+h)_k\, k!} \quad \left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y}\right). \tag{1.6}$$

We now introduce here the following multivariable analogues of the Burchnall–Chaundy symbolic operators $\nabla_{xy}(h)$ and $\Delta_{xy}(h)$ defined by (1.5) and (1.6), respectively (*cf.* [9, p. 240]; see also [10, p. 113] for the case when $r = 3$):

$$\tilde{\nabla}_{x_1;x_2\cdots x_r}(h) := \frac{\Gamma(h)\,\Gamma(\delta_1+\cdots+\delta_r+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\cdots+\delta_r+h)}$$
$$= \sum_{k_2,\ldots,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+\cdots+k_r} (-\delta_2)_{k_2} \cdots (-\delta_r)_{k_r}}{(h)_{k_2+\cdots+k_r}\, k_2! \cdots k_r!} \tag{1.7}$$

and

$$\tilde{\Delta}_{x_1;x_2\cdots x_r}(h) := \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\cdots+\delta_r+h)}{\Gamma(h)\,\Gamma(\delta_1+\cdots+\delta_r+h)}$$
$$= \sum_{k_2,\ldots,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+\cdots+k_r} (-\delta_2)_{k_2} \cdots (-\delta_r)_{k_r}}{(1-h-\delta_1-\cdots-\delta_r)_{k_2+\cdots+k_r}\, k_2! \cdots k_r!}$$