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A robust adaptive method for singularly perturbed convection–diffusion problem with two small parameters[☆]

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ABSTRACT

Numerical treatment for a one dimensional two-parameter singularly perturbed convection–diffusion equation is proposed and analyzed. The upwind finite difference method is applied to the problem. Maximum-norm a posteriori error estimates are obtained and used to establish an adaptive algorithm. The uniform convergence of the new algorithm is proved under feasible conditions of two parameters. Numerical experiments are given to confirm the analytical results.

1. Introduction

Consider the problem

$$\begin{cases} Lu = -\varepsilon_1 u''(x) - \varepsilon_2 b(x)u'(x) + c(x)u(x) = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \varepsilon_1,\ \varepsilon_2 \ll 1$. Suppose that the functions $b(x)$, $c(x)$ and $f(x)$ are all in $C^1[0, 1]$, and $b(x)$ and $c(x)$ satisfy especially

$$0 < \beta \le b(x) \le \overline{\beta},\ 0 < \gamma \le c(x),\ \forall x \in [0, 1], \quad \rho = \min_{x\in[0,1]} \frac{c(x)}{b(x)}. \tag{1.2}$$

The presence of the two small parameters ε_1, ε_2 results in difficulties of the numerical treatment of the solution on the boundary layers at the two boundaries $x = 0$ and $x = 1$, for problem (1.1). Many research papers have dedicated to the numerical treatment for the singularly perturbed problem. One can refer to [1] for an overview. Many numerical methods applied to the singularly perturbed problem to obtain an accurate numerical solutions are based on the use of the special meshes which are very fine in the layer region of the solution. Such meshes can be divided into two classes: the one constructed a priori, and the another constructed a posteriori. The mesh constructed a priori needs a priori information about the structure of the solution, such as the location and the width of the layer and so on. Two most well-known layer-adapted

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meshes constructed a priori are the Shishkin mesh and the Bakhvalov mesh (see also in [1–4]). However, in many situations, researchers discover that the priori information is unavailable; alternatively, they turn to the posteriori mesh in this case.

The posteriori mesh is constructed based on a posteriori error analysis. Kopteva pioneered in maximum norm a posteriori error estimates both for semilinear convection–diffusion problems in space dimension one [5] and semilinear reaction–diffusion problems in space dimension one to three [6–8]. After Kopteva, a series of new results in maximum norm a posteriori error estimates are done by many researchers appeared for a broad range of problems. For example, Chen in [9] considered a singularly perturbed problem in non-conservative form. An adaptive mesh based on maximum norm a posteriori error analysis was constructed with uniform $O(N^{-1})$ convergence by the author. In [10] Sun et al. considered a convection–diffusion–reaction problem. They derived maximum norm a posteriori estimates and designed an adaptive algorithm which is proved to be $O(N^{-1})$. But how about the techniques of the maximum norm a posteriori estimates for a singularly perturbed problem with two small parameters? One found that Linss' work in [11] used the streamline diffusion finite element methods(SDFEM) to solve the two-parameter problem. The author derived a robust a posteriori error estimate in the maximum norm and then constructed an adaptive algorithm but made no analysis about the adaptive algorithm. In the present paper, we will prove a robust maximum norm a posteriori error estimate for the upwind finite difference methods applied to the singularly perturbed problem with two small parameters and design an adaptive algorithm. We analyze the proposed adaptive algorithm in a rigorous way, and prove that, with the restriction of $\beta\varepsilon_2^2 \geq \rho\varepsilon_1$, the algorithm is uniformly convergent with the rate $O(N^{-1})$ under some reasonable assumptions, where the parameters β, ρ are defined in (1.2).

The paper is organized as follows. In Section 2, stability properties of the differential operator are considered. In Section 3, the discretization is applied to the problem. Maximum norm a posteriori error estimates are derived as well for the discrete scheme on an arbitrary mesh. In Section 4, an adaptive algorithm is designed based on the equidistribution associated with maximum norm estimates obtained in the previous section. Then, the convergence is given after the error analysis of the full-discrete scheme. Section 5 is devoted to the numerical experiments which confirm the theoretical analysis in Section 4 and illustrate the numerical performance of the adaptive algorithm.

Notation. Throughout the paper, we use C or C_* to denote generic positive constants independent of ε_1, ε_2, N and the number of the iterations taken by the adaptive algorithm described in Section 4, which may take different values in different situations.

2. Stability properties of differential operator

The solution of problem (1.1) is closely related to the solutions of the following characteristic equation

$$-\varepsilon_1\lambda(x)^2 - \varepsilon_2 b(x)\lambda(x) + c(x) = 0. \tag{2.1}$$

The two solutions of the characteristic equation are

$$\begin{cases} \lambda_0(x) = -\dfrac{1}{2\varepsilon_1}\left(\varepsilon_2 b(x) + \sqrt{4\varepsilon_1 c(x) + \varepsilon_2^2 b(x)^2}\right), \\ \lambda_1(x) = -\dfrac{1}{2\varepsilon_1}\left(\varepsilon_2 b(x) - \sqrt{4\varepsilon_1 c(x) + \varepsilon_2^2 b(x)^2}\right), \end{cases} \tag{2.2}$$

where $\lambda_0 < 0$ characterizes the boundary layer at $x = 0$, while $\lambda_1 > 0$ characterizes the boundary layer at $x = 1$. See [11, 12] for details. For the sake of simplicity, we set

$$\mu_0 := \max_{x\in[0,1]} \lambda_0(x) < 0, \qquad \mu_1 := \min_{x\in[0,1]} \lambda_1(x) > 0. \tag{2.3}$$

Remark 2.1 (*[12]*). Note that $|\lambda_0(x)| > \lambda_1(x)$ means a very sharp change of the solution near $x = 0$. It is said usually that the layer at $x = 0$ is stronger than that at $x = 1$.

2.1. Estimates of the Green's function

At first, it is easy to prove that problem (1.1) satisfies the comparison theorem, and the following stability inequality (cf. [1], for example)

$$\|u\|_\infty \leq \frac{1}{\gamma}\|f\|_\infty \tag{2.4}$$

holds.

Consider the Green's function $G(x, \xi)$ corresponding to the differential operator L. It is defined, for fixed $\xi \in (0, 1)$, to be the solution of the problem

$$\begin{cases} (LG(\cdot, \xi))(x) = \delta(x - \xi), \quad x \in (0, 1), \\ G(0, \xi) = G(1, \xi) = 0. \end{cases} \tag{2.5}$$