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Study of linear isotropic micro-polar plate in an asymptotic approach

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ABSTRACT

Linear micro-polar plate theory in an asymptotic approach is a two-dimensional approximation of three dimensional Cosserat theory. It was originally developed by Eringen. The current paper deals with the study of this type of plate theory with some modifications and an attempt has been made to develop a higher-order plate theory, especially third-order plate theory and to express the solution of the plate equations under that theory as a function of the solution of the equations of first-order micro-polar plate theory, which have already been well-discussed.

1. Introduction

In the year 1896, the Cosserat brothers, Eugén and François, published a monograph [1,2], where they presented a new variant of continuum mechanics as well as the mechanics of rods and shells. They have attempted to unify field theories of mechanics, optics and electrodynamics through a common principle of least action (Euclidean action). They found the appropriate general form of energy for variational problems. They postulated that the invariance of energy under Euclidean transformation was able to derive the balance of force and balance of momentum. However, they never wrote down any constitutive equations. This continuum model was later named the Cosserat or micro-polar continuum. The Cosserat model is one of the most prominent extended continuum models and features three additional independent degrees of freedom which are related to the rotation of particle, and they need not coincide with the macroscopic rotation of continuum.

The classical continuum considers a material as simple point-continua with points having three displacement degrees of freedom, and the response of the material to the displacement of its points is characterized by a symmetric Cauchy-stress tensor assuming that the transmission of loads through surface elements is uniquely determined by a force vector, neglecting the couples. Such a model may be insufficient for the description of certain physical phenomenon. It is considered that the material behaves in a non-classical way due to the micro-structural effects which are observed mostly in regions of high strain-gradient.

One of the initiators of axiomatic approaches in Cosserat's idea was W. Noll [3]. In micro-polar theory, coupled stresses are introduced [4] in addition to ordinary stresses. The deformation of a micro-polar continuum may be described by the position vector and three orthonormal vectors, so-called rigid directors, which model the translation and the orientation changes of material particles. The linear micro-polar theory was developed in the original papers by Günther [5], Aero and Kuvshinskii [6,7], Toupin [8,9], Mindlin and Tiersten [10], Koiter [11], Pal'mov [12], Eringen [13], Schaefer [14], Ieşan [15], etc. Many references to other papers can be found from the books [16–18].

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The main problem of any micro-polar theory is the establishment of a constitutive equation and this problem was not discussed in Cosserat's original monograph [1]. A difficulty arises in identifying the material parameter even when the problem is solved in a satisfactory manner. In the linear micro-polar elasticity of isotropic solids one needs six materials parameters while only two Lamé moduli are needed in the classical elasticity. The experimental identification of elastic moduli is discussed in [19–22]. R. Lakes proposed an experimental procedure to determine four additional material moduli of a linear micro-polar isotropic continuum, but is not always achievable for all heterogeneous materials in reality. Many relevant discussions on this matter are available on the websites by P. Neff and R. Lakes. There are other problems in determining the parameters of linear micro-polar theory due to size-effects, that means the small specimens behave in a relatively stiffer manner than the larger specimens of the same materials. The size-effect is an experimental fact which makes material parameters size-dependent. So this theory is inconsistent in describing size-effects in a continuous medium. However in the dynamical problem of elasticity involving elastic vibrations of high frequencies and short wavelength, the linear micro-polar theory is successful in reducing the difference between theoretical result and experimental facts which occurs by using classical linear theory of elasticity. The reason for such differences lies in the micro-structure of the material which exerts a special influence. Linear dynamic micro-polar theory predicts dispersion relations, which is impossible in classical linear elasticity though an experimental fact. This theory also predicts rotational waves which have not been observed experimentally.

Since the paper of Ericksen and Truesdell [23], the Cosserat model has found applications in the construction of various generalized models for beams, plates, and shells. This approach is developed in the original papers of Ericksen [24,25], Green and Naghdi [26], and Naghdi [27]. Fundamental books by Naghdi [28] and Rubin [29] are well-referred in the literature where the theory of the Cosserat shell is presented.

Classical plate theory really developed after the pioneering work of Kirchhoff. After that thousands of publications were presented which try to give the foundations and methods of deduction of the Kirchhoff–Love theory and its possible improvements. Books from Ciarlet [30,31] can be mentioned in this context. The assumptions on which the theory of small deflection of a thin elastic plate is based can be found in [32]. One of the methods which has been used to obtain a two dimensional model of thin elastic plates is the so-called asymptotic expansion method. In this method, a formal power series expansion of three-dimensional solution is used by considering the thickness of the plate as the small parameter and the Kirchhoff model of linear elastic isotropic plates is obtained as the leading term of formal asymptotic expansion. The early works of Goldenveizer [33], Friedrichs and Dressler [34] etc., are representative examples of this approach. A rigorous mathematical reformulation of the asymptotic approach has been given by Ciarlet and Destuynder [35] in which the three-dimensional problem is posed in variational form and a functional framework is used. There is another popular classical theory by taking into account shear effect that is ignored in Kirchhoff–Love plate theory, which is known as Reissner–Mindlin plate theory.

The Cosserat model can be posed in a variational format as a two-field minimization problem for usual displacement $\vec{u}$ and infinitesimal micro-rotation $\vec{\phi}$, which is an element of Lie-algebra SO(3). The application of such an approach in the development of plate theory is discussed in the current papers of Neff [36] and Altenbach and Eremeyev [37].

Independently, Eringen has formulated a linear theory of micro-polar plates in [38]. The two-dimensional equations of this theory are deduced with the help of independent integration over thickness. Theories of zeroth-order and first-order are presented by applying a special linear approximation of the displacement and micro-rotation field. Eringen's theory is based on eight unknowns: the averaged displacement, averaged macro-rotation of the cross-sections and averaged micro-rotation. Like in the Love–Kirchhoff assumption in the derivation of classical plate theory, a priori assumptions regarding the variation of unknowns across the thickness of the plate are made which are given as follows:

i. The transverse displacement component is independent of the transverse coordinate. The in-plane displacement components are not only functions of the in-plane coordinates but they are also linear functions of transverse coordinates.
ii. Both transverse and in-plane components of micro-rotation are independent of transverse coordinates.

Green and Naghdi [26] have derived a different set of micro-polar plate equations by using a method of asymptotic expansion. They have discussed the micro-polar and director theories of plates and compared these to a theory derived for a Cosserat plate. They concluded that the method used by Eringen does not provide a consistent approximate theory of plates based on three-dimensional equations of micro-polar elasticity. Attempts have been made to incorporate terms of second approximation for improvement of micro-polar plate theory that was initiated by Eringen, which can be seen from the work of Erbay [39]. More references and various aspects of micro-polar plate theories of Eringen can be seen from the paper of Wang [40].

In this paper we have presented a linear micro-polar plate theory in an asymptotic approach by incorporating higher order terms with further modification of Eringen's approach from a theoretical point of view. We have attempted to make a relationship between the solution of the second order theory and the solution of the first-order theory and discussed the uniqueness of the problem by considering boundary conditions in a special way.

2. Micro-polar linear elasticity

The theory of linear micro-polar elasticity was discussed and developed through the works of Nowacki [18], Eringen [17] and Ieşan [15], etc. In this present section, we present some basic relations of the linear micro-polar elasticity which is based on the fundamental works of Eringen.