Computers and Mathematics with Applications 71 (2016) 76–84

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Towards optimal finite element error estimates for the penalized Dirichlet problem in a domain with curved boundary

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ARTICLE INFO

Article history:
Received 3 July 2015
Received in revised form 29 October 2015
Accepted 30 October 2015
Available online 21 November 2015

Keywords:
Elliptic equations
Curved boundary
Finite element
Dirichlet boundary conditions
Penalty method

ABSTRACT

We consider the finite element approximation of an elliptic problem with homogeneous Dirichlet boundary conditions on a curved boundary and imposed using the penalty method. We establish optimal H^1 error estimates with suitable assumptions on the penalty parameter ε as a function of the elements size h. Our focus is on establishing these results with least restrictive assumptions possible on this dependency.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^d (d = 2, 3)$ with boundary $\partial\Omega$ and let $f \in L^2(\Omega)$. We consider the elliptic equation

$$-\Delta u + u = f, \quad \text{in } \Omega, \tag{1}$$

with the homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial\Omega. \tag{2}$$

If $\partial\Omega$ is Lipschitz continuous ($\mathscr{C}^{0,1}$) then this problem has a unique weak solution $u \in H_0^1(\Omega)$ satisfying

$$A(u, v) = F(v), \quad \forall\, v \in H_0^1(\Omega), \tag{3}$$

where the bilinear form A and the linear form F are defined by

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u\, v\, dx,$$
$$F(v) := \int_\Omega f\, v\, dx.$$

Due to the symmetry of A, u is also the minimizer of the quadratic functional $J(v) := \frac{1}{2}A(v, v) - F(v)$ over $H_0^1(\Omega)$.

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http://dx.doi.org/10.1016/j.camwa.2015.10.019

Instead of working in the constrained set $H_0^1(\Omega)$ induced by the homogeneous Dirichlet boundary condition, we use the penalty method to relax this constraint and to work on the entire space $H^1(\Omega)$. A common penalized formulation is then: Given $\varepsilon > 0$, find $u_\varepsilon \in H^1(\Omega)$, such that

$$A_\varepsilon(u_\varepsilon, v) = F(v), \quad \forall\, v \in H^1(\Omega), \tag{4}$$

where

$$A_\varepsilon(u_\varepsilon, v) := A(u_\varepsilon, v) + \frac{1}{\varepsilon}\int_{\partial\Omega} u_\varepsilon\, v\, ds.$$

The unique solution u_ε is also the minimizer of the perturbed quadratic functional $J_\varepsilon(v) := J(v) + \frac{1}{2\varepsilon}\int_{\partial\Omega} |v|^2 ds$ and Eq. (4) is a weak (or variational) form of the Robin type boundary value problem

$$-\Delta u_\varepsilon + u_\varepsilon = f, \quad \text{in } \Omega, \tag{5}$$

$$\nabla u_\varepsilon \cdot \boldsymbol{n} + \frac{1}{\varepsilon} u_\varepsilon = 0, \quad \text{on } \partial\Omega. \tag{6}$$

The *penalized solution* u_ε is expected to converge to u when the penalty parameter ε goes to 0. If $\partial\Omega$ is sufficiently smooth, for instance $\mathscr{C}^{1,1}$, then a general result of Maury [1] (see Theorem 1 below) implies that $\|u - u_\varepsilon\|_{H^1(\Omega)} = \mathcal{O}(\varepsilon)$ if u is regular enough, more precisely in $H^2(\Omega)$. If $\partial\Omega$ is only Lipschitz continuous ($\mathscr{C}^{0,1}$), like with a polygonal domain, then the general result of [1] does not apply, unless the normal derivative $\partial u/\partial n$ is assumed to lie in $H^{1/2}(\partial\Omega)$, a condition that does not hold generally if $u \in H^2(\Omega)$ due to the low regularity of $\partial\Omega$. To the best of our knowledge, assuming only $u \in H^2(\Omega)$, the best theoretical convergence result is $\|u - u_\varepsilon\|_{H^1(\Omega)} = \mathcal{O}(\varepsilon^{1/2})$. This can be proved as in [2] for the (elliptic) Lamé system of equations with ideal contact boundary conditions which include a Dirichlet-type boundary condition.

Here we consider penalty-finite element approximations of the original problem (3) which are finite element approximations of the penalized problem (4). Let h denote the discretization parameter, that is the size of the elements constituting the regular mesh of Ω or a polyhedral approximation Ω_h of Ω. We consider finite element approximation spaces V_h made of continuous piecewise polynomials of degree k over Ω or Ω_h respectively. Then, choosing the penalty parameter in the form $\varepsilon = h^\lambda$ with a suitable value of $\lambda > 0$, we expect the convergence to be optimal with respect to the $H^1(\Omega)$-norm: $\|u - u_{\varepsilon,h}\|_{H^1(\Omega)} = \mathcal{O}(h^k)$.

Obtaining such optimal convergence estimates is not so easy and is not even proven in all cases. Take for instance the case where Ω is polyhedral (or polygonal), in which case it can be exactly partitioned into a union of (for instance) tetrahedral elements and the resulting approximation is conforming ($V_h \subset H^1(\Omega)$). A standard estimate (see Exercise 3.2.2 in Ciarlet [3] or Proposition 2.10 in Maury [1]) is of the form

$$\|u - u_{\varepsilon,h}\|_{H^1(\Omega)} \le C\left(\frac{h^k}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\right),$$

if $u \in H^{k+1}(\Omega)$. Unfortunately, this estimate cannot provide an optimal convergence rate because of the presence of a negative power of ε in the right hand side which is due to the continuity constant of the bilinear form A_ε which is not uniform with respect to ε as $\varepsilon \to +\infty$. However, another simple argument given in [1], Proposition 2.9, leads to

$$\|u - u_{\varepsilon,h}\|_{H^1(\Omega)} \le C(h^k + \sqrt{\|u - u_\varepsilon\|_{H^1(\Omega)}}),$$

if $u \in H^{k+1}(\Omega)$, which thus gives the bound $C(h^k + \varepsilon^{1/4})$ according to what we mentioned previously in the case of a polyhedral domain Ω, if we do not assume the extra regularity $\partial u/\partial n \in H^{1/2}(\partial\Omega)$. This estimate cannot be seen as optimal in terms of ε, since in the limit case $h = 0$ it only gives an $\mathcal{O}(\varepsilon^{1/4})$ convergence. Nevertheless, it shows that an optimal convergence rate $\mathcal{O}(h^k)$ is achieved if $\lambda \ge 4k$. To the best of our knowledge, the best estimates were obtained by Barrett and Elliott [4] who proved the convergence rate to be optimal with $\lambda \ge k + 1/2$ and $u \in H^{k+1}(\Omega)$. Let us also add that the results in [4] give an optimal $\mathcal{O}(h^{k+1})$ rate in the $L^2(\Omega)$-norm, but with the stronger assumption that $\lambda \ge k + 1$ and $u \in H^{k+2}(\Omega)$.

In the present work we consider a smoother (curved) boundary $\partial\Omega$, at least $\mathscr{C}^{1,1}$. Two classes of tetrahedral meshes were considered in the literature on the penalty method. For the first class, $\Omega \subset D_h$, the union of tetrahedral elements, and the integrations involved in the variational formulation are performed exactly over Ω and $\partial\Omega$. This was first considered by Babuska [5] and his results were later improved by Barrett and Elliott [4] who obtained optimal H^1 convergence rates for $k = 1$ and $k = 2$ with, respectively, $1 \le \lambda \le 2$ and $\lambda = 2$, under the assumption that $u \in H^{k+2}(\Omega)$. For the second class of meshes, an approximation Ω_h of Ω is meshed and the integrations involved in the (approximate) variational formulation are performed over Ω_h and $\partial\Omega_h$. Barrett and Elliot only considered linear elements ($k = 1$) and showed an optimal H^1 convergence rate with the only choice $\lambda = 2$, under the assumption that $u \in H^4(\Omega)$.

In this paper we consider the latter case, where integrations are performed over Ω_h, a case of variational crime ($V_h \not\subseteq H^1(\Omega)$). Our goal is to obtain optimal H^1 convergence rates for every choice of k and with less restrictions on λ, in particular with no upper bound for λ. Circumventing such a restriction can assure the practitioner that, for a given mesh (that is for a