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On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields

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ABSTRACT

We propose sharp results on the numerical approximation of low-regularity electromagnetic fields by edge finite elements. We consider general geometrical settings, including topologically non-trivial domains or domains with a non-connected boundary. In the model, the electric permittivity and magnetic permeability are symmetric, tensor-valued, piecewise smooth coefficients. In all cases, the error can be bounded by h^δ times a constant, where h is the meshsize, for some exponent $\delta \in]0, 1]$ that depends both on the geometry and on the coefficients. It relies either on classical estimates when $\delta > 1/2$, or on a new combined interpolation operator when $\delta < 1/2$. The optimality of the value of δ is discussed with respect to abstract shift theorems. In some simple configurations, typically for scalar-valued permittivity and permeability, the value of δ can be further characterized. This paper is the first one in a series dealing with the approximation of electromagnetic fields by edge finite elements.

1. Introduction

The aim of this paper is to study the numerical approximation of electromagnetic fields, governed by Maxwell's equations with volume sources in bounded regions of $\mathbb{R}^3$. More precisely, we are interested in exhibiting the approximation capabilities of those fields with the help of edge element interpolation operators. Typically, the domain under scrutiny is bounded and enclosed in a perfect conductor, and it can be made of different materials. In particular, we shall provide interpolation results that depend on the geometry of the domain, on the electric permittivity and the magnetic permeability that describe the materials, and also on the regularity of the sources, that is the current and charge densities. Special attention will be devoted to cases where the regularity of the fields is minimal.

In the next section, we begin by recalling a model problem in electromagnetic theory, namely the time-harmonic Maxwell equations set in a bounded domain. We recall equivalent variational formulations and well-posedness results, and the approximation by edge elements. To obtain a priori convergence estimates, we then study the minimal regularity of those fields: this is the main topic of Section 3. The regularity results are derived thanks to a splitting of the fields and their curls into a regular part and a gradient. In Section 4, we study in detail the approximability by edge finite elements of the fields. We review the classical interpolation results, before we define a new, combined, interpolation operator which relies explicitly on the splitting of the fields, and not only on the minimal regularity. We conclude this section by a comparison with the

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more recent quasi-interpolation theory. As a result of the approximability properties, we finally derive in Section 5 optimal error estimates.

Throughout the paper, C is used to denote a generic positive constant which is independent of the meshsize, the triangulation and the fields of interest. On the other hand, C may depend on the geometry of the domain, or on the coefficients defining the model. We also use the shorthand notation $A \lesssim B$ for the inequality $A \le CB$, where A and B are two scalar fields, and C is a generic constant. Respectively, $A \approx B$ for the inequalities $A \lesssim B$ and $B \lesssim A$. We denote constant fields by the symbol *cst*. Vector-valued (resp. tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters); for the latter, the index *sym* indicates symmetric fields. Given an open set $\mathcal{O}$ of $\mathbb{R}^3$, we use the notation $(\cdot|\cdot)_{0,\mathcal{O}}$ (respectively $\|\cdot\|_{0,\mathcal{O}}$) for the $L^2(\mathcal{O})$ and the $\boldsymbol{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$ hermitian scalar products (resp. norms). More generally, $(\cdot|\cdot)_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ (respectively $|\cdot|_{s,\mathcal{O}}$) denote the hermitian scalar product and the norm (resp. semi-norm) of the Sobolev spaces $H^s(\mathcal{O})$ and $\boldsymbol{H}^s(\mathcal{O}) := (H^s(\mathcal{O}))^3$ for $s \in \mathbb{R}$ (resp. for $s > 0$). The index *zmv* indicates zero-mean-value fields. If moreover the boundary $\partial\mathcal{O}$ is Lipschitz, $\boldsymbol{n}$ denotes the unit outward normal vector field to $\partial\mathcal{O}$. Finally, it is assumed that the reader is familiar with function spaces related to Maxwell's equations, such as $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}_0(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}(\mathrm{div}\,;\mathcal{O})$, and $\boldsymbol{H}_0(\mathrm{div}\,;\mathcal{O})$. We refer to the monograph of Monk [1] for details. We will define more specialized function spaces later on.

2. Time-harmonic problems in electromagnetics

Let Ω be a domain in $\mathbb{R}^3$, i.e. an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. For a given pulsation $\omega > 0$, the time-harmonic Maxwell's equations (with time-dependence $\exp(-\imath\omega t)$) write

$$\mathbf{curl}\,\boldsymbol{h} + \imath\omega\varepsilon\boldsymbol{e} = \boldsymbol{j} \quad \text{in } \Omega, \tag{1}$$

$$\mathbf{curl}\,\boldsymbol{e} - \imath\omega\mu\boldsymbol{h} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathrm{div}\,\varepsilon\boldsymbol{e} = \varrho \quad \text{in } \Omega, \tag{3}$$

$$\mathrm{div}\,\mu\boldsymbol{h} = 0 \quad \text{in } \Omega. \tag{4}$$

Above, the real-valued coefficient ε is the electric permittivity tensor and the real-valued coefficient μ is the magnetic permeability tensor, whereas $(\boldsymbol{e}, \boldsymbol{h})$ is the couple of electromagnetic fields, and the source terms $\boldsymbol{j}$ and ϱ are respectively the current density and the charge density. The latter are related by the charge conservation equation

$$-\imath\omega\varrho + \mathrm{div}\,\boldsymbol{j} = 0 \quad \text{in } \Omega. \tag{5}$$

The other two electromagnetic fields are the electric displacement $\boldsymbol{d}$ and the magnetic induction $\boldsymbol{b}$. They are related to $\boldsymbol{e}$ and $\boldsymbol{h}$ by the constitutive relations

$$\boldsymbol{d} = \varepsilon\boldsymbol{e}, \qquad \boldsymbol{b} = \mu\boldsymbol{h} \quad \text{in } \Omega. \tag{6}$$

In what follows, we focus mainly on the couple of fields $(\boldsymbol{e}, \boldsymbol{h})$. However the results are easily extended to the couple of fields $(\boldsymbol{d}, \boldsymbol{b})$ thanks to the relations (6).

We assume that the coefficients ε, μ, together with their inverses ε^{-1}, μ^{-1}, belong to $\mathbb{L}^\infty_{sym}(\Omega)$. Classically,[1] to be able to define the electromagnetic energy, they are such that $\lambda_{\min}(\varepsilon) > 0$ and $\lambda_{\min}(\mu) > 0$ a.e. in Ω where $\lambda_{\min}$ stands for the smallest eigenvalue, and the couple of electromagnetic fields belongs to $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$. We choose source terms $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$, and $\varrho \in H^{-1}(\Omega)$.

We assume that the medium Ω is surrounded by a perfect conductor, so that the boundary condition below holds:

$$\boldsymbol{e} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Hence the couple of electromagnetic fields $(\boldsymbol{e}, \boldsymbol{h})$ belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathbf{curl};\Omega)$.

2.1. Variational formulations

The Maxwell problem can be formulated in the electric field $\boldsymbol{e}$ only, namely

$$\begin{cases} \text{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ such that} \\ -\omega^2\varepsilon\boldsymbol{e} + \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{e}) = \imath\omega\boldsymbol{j} \quad \text{in } \Omega \\ \mathrm{div}\,\varepsilon\boldsymbol{e} = \varrho \quad \text{in } \Omega. \end{cases} \tag{8}$$

Note that in (8), the equation $\mathrm{div}\,\varepsilon\boldsymbol{e} = \varrho$ is implied by the second-order equation $-\omega^2\varepsilon\boldsymbol{e} + \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{e}) = \imath\omega\boldsymbol{j}$, together with the charge conservation Eq. (5), so it can be omitted. Furthermore, the magnetic field can be recovered using Faraday's law (2). Moreover, one can check that the equivalent variational formulation in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ writes

$$\begin{cases} \text{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ such that} \\ (\mu^{-1}\mathbf{curl}\,\boldsymbol{e}|\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2(\varepsilon\boldsymbol{e}|\boldsymbol{v})_{0,\Omega} = \imath\omega(\boldsymbol{j}|\boldsymbol{v})_{0,\Omega}, \quad \forall\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega). \end{cases} \tag{9}$$

[1] For more "exotic" configurations of Maxwell's equations, in which ε or μ exhibit a sign-change of one or several eigenvalues across some interface, we refer to [2–4].