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# Efficient numerical schemes for fractional water wave models

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ABSTRACT

In this paper, efficient numerical schemes are proposed for solving the fractional water wave models that describe the propagation of surface water wave. By using the weighted and shifted Grünwald–Letnikov (WSGL) formula to approximate the nonlocal fractional operators, we design a series of second order accurate difference schemes for the considered models. The existence, stability and convergence of numerical solutions of the proposed numerical schemes are established rigorously. The analysis shows that the proposed numerical schemes are unconditionally stable with second order accuracy for both temporal and spatial discretizations. Several numerical results are provided to verify the efficiency and accuracy of our theoretical analysis.



## 1. Introduction

In this paper, we consider the efficient numerical schemes for solving the nonlocal water wave model

$$u_t + f(u)_x - \alpha u_{txx} + \mathcal{D}_x^{\mu} u = \nu u_{xx}, \tag{1}$$

where $u = u(x,t)$ represents the vertical displacement of the surface of the wave from its equilibrium, $x$ is proportional to distance in the direction of wave and $t$ is proportional to elapsed time. $\mathcal{D}_x^{\mu}$ is the linear combination of left and right Riemann–Liouville fractional derivatives

$$\mathcal{D}_x^{\mu} u(x,t) = \left(\kappa_1\, {}_aD_x^{\mu} + \kappa_2\, {}_xD_b^{\mu}\right) u(x,t), \quad 0 < \mu < 1, \tag{2}$$

with ${}_aD_x^{\mu}$ and ${}_xD_b^{\mu}$ being left and right Riemann–Liouville fractional derivatives of order $\mu$, respectively, defined by [1]

$${}_aD_x^{\mu} u(x,t) = \frac{1}{\Gamma(1-\mu)} \frac{\partial}{\partial x} \int_a^x (x-s)^{-\mu} u(s,t)\,ds, \tag{3}$$

and

$${}_xD_b^{\mu} u(x,t) = \frac{-1}{\Gamma(1-\mu)} \frac{\partial}{\partial x} \int_x^b (s-x)^{-\mu} u(s,t)\,ds. \tag{4}$$

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Here the parameters $\alpha$, $\nu$ and $\kappa_1, \kappa_2$ are nonnegative constants which describe the balance, the effects of viscosity and dispersion. And the Fourier symbol of the operator of the fractional derivative defined in (2) gives [2,3]

$$\mathcal{F}(\mathcal{D}_x^{\mu}) = \kappa_1(ik)^{\mu} + \kappa_2(-ik)^{\mu} = \left[\cos\left(\frac{\pi\mu}{2}\right) + i(\kappa_1 - \kappa_2)\sin\left(\frac{\pi\mu}{2}\right)\right]|k|^{\mu}, \quad k \in \mathbb{R},\ i^2 = -1. \tag{5}$$

Equations of the form (1) appear in many models concerning the propagation of small-amplitude, nonlinear, dispersive, dissipative wave equations, see [4,5] and references therein. The evolution Eq. (1) includes several classical equations as special cases. For example, if $f(u) = u + \delta\frac{u^2}{2}, \delta > 0, \nu = 0, \mu = \frac{1}{2}$, the model reduces to fractional Kakutani–Matsuuchi model [5,6]

$$u_t + u_x - \alpha u_{txx} + \mathcal{D}_x^{\frac{1}{2}} u + \delta u u_x = 0. \tag{6}$$

If $\alpha = 0, f(u) = \frac{u^2}{2}$, the model reduces to the fractional Fowler equation [7,8]

$$u_t + u u_x + \mathcal{D}_x^{\mu} u = \nu u_{xx}. \tag{7}$$

And if $\nu = 0, f(u) = 0, \mu = \frac{1}{2}$, the model reduces to the fractional water wave equation [9]

$$u_t + \mathcal{D}_x^{\frac{1}{2}} u = 0. \tag{8}$$

The above mentioned models are Boussinesq systems with a nonlocal viscous term, they play more and more important role in investing the effect of viscosity on the gravity wave. In the previous studies, theoretical analysis for the nonlinear differential equations with a nonlocal term has been investigated [10–14]. For the well-posedness, regularity and asymptotic behaviors of solutions of the Cauchy problem for (6) with fractional Laplace operator (5), one can see [4,5]. Many different numerical methods, including some high order and fast algorithms [15–26], have been developed in the literature for the computation of linear fractional differential equations. In contrast to the linear problem, there are not many works on numerical methods for the nonlinear partial differential equations involving the fractional derivatives. A brief overview of the numerical studies for nonlinear fractional differential equations are given below. Biler et al. [27] developed a numerical method based on the interacting particles approximation for the solution of a large class of evolution problems involving the fractional Laplacian and a non-local quadratic-type non-linearity. Ervin et al. [28] developed a fully finite element approximation to a time dependent fractional diffusion equation which contains a nonlocal quadratic nonlinearity. Recently, Droniou [10] constructed a class of finite difference schemes with one order for the fractal conservation laws. It is proved that the numerical solutions converge toward Alibaud's entropy solution. The discontinuous Galerkin approximation of nonlinear conservation law with fractional Laplace operator has been discussed and a Kuznetsov type of theory has been established and applied to obtain the error estimates, see [29]. A Runge–Kutta local discontinuous Galerkin method has been proposed and stability and error estimations were derived for nonlinear conservation law with fractional Laplace operator in [30] by Xu and Hesthaven. In Chen's recent work [5], a Fourier spectral approximation was used to capture the time decay behavior of Eq. (6). Jennings discussed some efficient numerical methods for the fractional water wave equation (8) in unbounded domains with nonreflecting boundaries [9]. More recently, a semi-implicit spectral defect correction method is constructed for a nonlocal Kakutani–Matsuuchi model [31].

The main purpose of this work is to develop and investigate some effective difference techniques for fractional water waves model (1) on a finite domain. With the help of the weighted and shifted Grünwald–Letnikov (WSGL) formula, which was originally developed in Ref. [18] to approximate the nonlocal fractional operators, we design a series of second order accurate difference schemes for the model (1) with the linear and nonlinear convection terms. Moreover, we prove the existence and the uniqueness of the solution for the proposed schemes and study the properties of the numerical solutions. We show that our schemes are unconditionally convergent with second-order accuracy in both temporal and space accuracy. To our best acknowledge, stable and second order accurate difference methods have never been constructed before in the literature for solving the model equation (1).

The rest of the paper is organized as follows. In Section 2, we introduce some notations and then briefly review the second difference discretizations of fractional derivatives. In Section 3, we first design numerical scheme for model (1) with linear convection, i.e., $f(u) = u$. Later, we discuss the associated stability estimates and convergence analysis of the presented difference schemes. Then, we discretize the nonlinear equation (1) with nonlinear convection $f(u) = u^2/2$. We discretize the model (1) in time via a second-order Crank–Nicolson-differentiation formula. The nonlinear term in Eq. (1) is treated by linearization in our algorithm so that the linear iteration is solved at each time-step. The existence, stability and error estimates of our presented numerical schemes are established. We demonstrate the desired performance of the proposed numerical schemes in Section 5 by extensive numerical examples. The numerical results reported in this section are in good agreement with the theoretical estimates and thus demonstrate the effectiveness of the proposed method. And we examine the effect of different parameters appearing in model (1). The numerical simulations show that the value of $\mu$ has significant effect on the profiles of the solutions of the presented models. Finally, we give our conclusion in Section 6.