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Finite element approximation of optimal control problems governed by time fractional diffusion equation

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ABSTRACT

In this paper Galerkin finite element approximation of optimal control problems governed by time fractional diffusion equations is investigated. Piecewise linear polynomials are used to approximate the state and adjoint state, while the control is discretized by variational discretization method. A priori error estimates for the semi-discrete approximations of the state, adjoint state and control are derived. Furthermore, we also discuss the fully discrete scheme for the control problems. A finite difference method developed in Lin and Xu (2007) is used to discretize the time fractional derivative. Fully discrete first order optimality condition is developed based on 'first discretize, then optimize' approach. The stability and truncation error of the fully discrete scheme are analyzed. Numerical example is given to illustrate the theoretical findings.

1. Introduction

The primary interest of this paper is to discuss Galerkin finite element approximation of optimal control problems governed by time fractional diffusion equation. Let Ω be a bounded domain of $R^d (1 \leqslant d \leqslant 3)$ with sufficiently smooth boundary $\partial\Omega$. Set $\Omega_T = \Omega \times (0, T)$, $\Gamma_T = \partial\Omega \times (0, T)$. We consider the following optimal control problems governed by time-fractional diffusion equations:

$$\min_{q \in U_{ad}} J(u, q) = \frac{1}{2}\|u(\mathbf{x}, t) - u_d(\mathbf{x}, t)\|^2_{L^2(\Omega_T)} + \frac{\gamma}{2}\|q(\mathbf{x}, t)\|^2_{L^2(\Omega_T)} \tag{1.1}$$

subject to

$$\begin{cases} {}_0\partial_t^\alpha u - \triangle u = f + q, & (\mathbf{x}, t) \in \Omega_T, \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Gamma_T, \\ u(\mathbf{x}, 0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{1.2}$$

Here ${}_0\partial_t^\alpha u$ denotes the left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of the state $u(\mathbf{x}, t)$. The other details will be specified later.

The motivation for this study comes from the fact that time fractional diffusion equations arise in many engineering applications such as anomalous diffusion on fractals and fractional random walk, see, for example, [1–4]. In the past decades

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the research for the numerical methods and algorithms of time fractional diffusion equations forms a hot topic. A number of numerical techniques were developed. We refer to [5–7] for finite difference method, [8] for implicit method, [9–12] for Galerkin finite element method, and [13–15] for finite difference method/spectral method. In [11] Galerkin finite element approximation of spatial fractional advection diffusion equations was discussed. A priori error estimate was proved for smooth solutions. However, in many practical problems, for example, inverse problems and optimal control problems, the solutions have only limited regularity. In [9] a priori error estimates for semi-discrete Galerkin finite element approximation of time fractional diffusion equations with limited regularity were investigated. For more details we refer the interested readers to the references cited herein.

Compared to optimal control problems with integer order derivative, there are few literature with respect to theoretical analysis or numerical methods devoted to optimal control problems governed by fractional diffusion equations. In [16] distributed optimal control problem governed by time fractional diffusion equations with Riemann–Liouville type time fractional derivative was considered. Existence, uniqueness and first order optimality conditions were proved. State constrained optimal control problem governed by time fractional diffusion equations was studied in [17]. In [18] optimal control problems governed by time fractional diffusion equations with delay was investigated. In [19] two numerical methods based on spectral method using Chebyshev polynomials were proposed for fractional optimal control problems governed by ordinary differential equations. A spectral approximation of inverse problems of reconstructing an initial value for a time fractional diffusion equation was studied in [20] on the basis of an optimal control framework without control constraints.

In the present paper we focus on Galerkin finite element approximation of optimal control problems (1.1)–(1.2). Piecewise linear polynomials are used to approximate the state and adjoint state, while the control is implicitly discretized by the variational discretization concept. For the time discretization we consider the finite difference method proposed in [13]. Regularity estimates for the state, adjoint state and the control are discussed. Fully discrete first order optimality condition is developed based on 'first discretize, then optimize' approach. A priori error estimate is proved for the semi-discrete case. The stability and truncation error of the fully discrete scheme are analyzed. Numerical example is presented to verify the theoretical findings.

Our paper is organized as follows. In the next section we present some preliminary knowledge to be used in the following sections. In Section 3 we investigate the existence and uniqueness of the optimal control problems, derive the first order optimality condition and prove some results with respect to the regularity of the solutions. In Section 4 we consider the semi-discrete Galerkin finite element approximation and prove a priori error estimates for the state variable, the adjoint state variable and the control variable, respectively. Fully discrete scheme is presented in Section 5. In Section 6 we present one numerical example to confirm our theoretical findings. Finally, we draw some concluding remarks in Section 7.

2. Preliminary

In this part we briefly recall some knowledge about Caputo, Riemann–Liouville fractional derivatives, fractional derivative spaces and Sobolev spaces.

For $0 < \alpha < 1$, the left Caputo and Riemann–Liouville time fractional derivatives are defined by

$$ {}_0\partial_t^\alpha v = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v'(s)}{(t-s)^\alpha} ds $$

and

$$ {}_0^R\partial_t^\alpha v = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds. $$

In a similar way the right Caputo and Riemann–Liouville fractional derivatives of order α are defined by

$$ {}_t\partial_T^\alpha v = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{v'(s)}{(s-t)^\alpha} ds $$

and

$$ {}_t^R\partial_T^\alpha v = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{v(s)}{(s-t)^\alpha} ds. $$

By a direct calculation [21] one has the following relation between Caputo and Riemann–Liouville fractional derivatives:

$$ {}_0^R\partial_t^\alpha v(t) = {}_0\partial_t^\alpha v + \frac{v(0)t^{-\alpha}}{\Gamma(1-\alpha)} $$

and

$$ {}_t^R\partial_T^\alpha v(t) = {}_t\partial_T^\alpha v + \frac{v(T)(T-t)^{-\alpha}}{\Gamma(1-\alpha)}. $$