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Eigenvalue clustering of coefficient matrices in the iterative stride reductions for linear systems

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ABSTRACT

Solvers for linear systems with tridiagonal coefficient matrices sometimes employ direct methods such as the Gauss elimination method or the cyclic reduction method. In each step of the cyclic reduction method, nonzero offdiagonal entries in the coefficient matrix move incrementally away from diagonal entries and eventually vanish. The steps of the cyclic reduction method are generalized as forms of the stride reduction method. For example, the 2-stride reduction method coincides with the 1st step of the cyclic reduction method which transforms tridiagonal linear systems into pentadiagonal systems. In this paper, we explain arbitrary-stride reduction for linear systems with coefficient matrices with three nonzero bands. We then show that arbitrary-stride reduction is equivalent to a combination of 2-stride reduction and row and column permutations. We thus clarify eigenvalue clustering of coefficient matrices in the step-by-step process of the stride reduction method. We also provide two examples verifying this property.

1. Introduction

Linear systems with tridiagonal coefficient matrices are utilized in solving ordinary differential equations [1], computing eigenvectors of a tridiagonal matrix [2], and generating spline functions [3]. The cyclic reduction method is a classical technique for solving linear systems with tridiagonal coefficient matrices [4,5]. This method involves a sequence of transformations of coefficient matrices such that two bands (except for the diagonal band and zero-values bands) move stepwise away from the diagonal band. In the final step of the cyclic reduction method, the coefficient matrix is transformed into a diagonal matrix. The cyclic reduction method has been extended to the block cyclic reduction method for application to block tridiagonal linear systems that appear when solving partial differential equations, such as Poisson's equation [6,7]. Block cyclic reduction, together with row and column permutations, enables the division of an original linear system into several smaller linear systems. The block cyclic reduction method is thus an important technique in parallel computing [8,9].

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Another extension of the cyclic reduction method is the stride reduction method. The $2M$-stride reduction method is applicable to linear systems with coefficient matrices of the form:

$$A := \begin{pmatrix} b_1 & & a_1 & & & \\ & \ddots & & \ddots & & \\ c_{M+1} & & b_{M+1} & & \ddots & \\ & \ddots & & \ddots & & a_{m-M} \\ & & \ddots & & \ddots & \\ & & & c_m & & b_m \end{pmatrix}. \tag{1}$$

In this paper, we refer to band matrix A as an M-tridiagonal matrix. A $2M$-tridiagonal matrix has the same form as that in (1), but replacing M with $2M$. These appear in coefficient matrices of linear systems following $2M$-stride reduction of linear systems with M-tridiagonal coefficient matrices. It is clear that 2-stride reduction is equal to the 1st step of the cyclic reduction method. Therefore, the 1st step of the cyclic reduction method may be regarded as a special case of stride reduction. This case is referred to hereinafter as 2-stride reduction.

In general, when solving linear systems, it is desirable to have eigenvalues of coefficient matrices be close to each other. Numerical difficulties in solving linear systems often depend on the ratios of the maximum and minimum eigenvalues of the coefficient matrices [10,11]. In symmetric matrices, these ratios are referred to as condition numbers. For simplicity, we hereinafter refer to the ratios of nonsymmetric matrices as extended condition numbers. Eigenvalues of matrices cluster as condition numbers (or extended condition numbers) become smaller. Extended condition numbers for pentadiagonal coefficient matrices following 2-stride reduction for some classes of tridiagonal linear systems have been described by Wang et al. [12]. If the original tridiagonal coefficient matrix is positive definite and has the same sign in each pair of symmetric subdiagonal entries, then the extended condition number of the resulting pentadiagonal coefficient matrix is smaller than that of the original tridiagonal matrix. However, the condition numbers and extended condition numbers of coefficient matrices have not yet been fully studied at all steps of the cyclic reduction method for obtaining a linear system with a diagonal coefficient matrix. Thus, the main purpose of this paper is to investigate condition numbers and extended condition numbers of coefficient matrices through the stride reduction method, including the cyclic reduction method.

The paper is organized as follows. In Section 2, we explain the arbitrary-stride reduction method. Next, in Section 3, we clarify the row and column permutations for transforming the M-tridiagonal matrix into a tridiagonal matrix without changing the eigenvalues of the matrix. In Section 4, we define arbitrary-stride reduction in terms of a combination of 2-stride reduction and row and column permutations. Using the findings of Wang et al. [12], in Section 5, we describe a remarkable property of coefficient matrices in linear systems involving their condition numbers or extended condition numbers following stride reduction, and then present two numerical examples. Finally, in Section 6, we give concluding remarks.

2. Arbitrary-stride reduction

The 2-stride, 3-stride and 4-stride reductions for tridiagonal linear systems are presented by Evans [4]. Thus, we will formulate the arbitrary-stride reduction along the same lines. It is easy to expand this reduction to $2M$-stride reduction for an arbitrary positive integer M. In this section, we define $2M$-stride reduction for transforming M-tridiagonal linear systems into $2M$-tridiagonal systems.

We consider the M-tridiagonal linear system $A\boldsymbol{x} = \boldsymbol{y}$ with a unique solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^\top \in \mathbf{R}^m$ and given $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^\top \in \mathbf{R}^m$. By focusing on the $(i-M)$th, ith and $(i+M)$th entries of both sides of $A\boldsymbol{x} = \boldsymbol{y}$, we easily derive the following:

$$c_{i-M}x_{i-2M} + b_{i-M}x_{i-M} + a_{i-M}x_i = y_{i-M}, \tag{2}$$
$$c_i x_{i-M} + b_i x_i + a_i x_{i+M} = y_i, \tag{3}$$
$$c_{i+M}x_i + b_{i+M}x_{i+M} + a_{i+M}x_{i+2M} = y_{i+M}. \tag{4}$$

By multiplying both sides of (2) by $-c_i/b_{i-M}$ and adding them to both sides of (3), we derive

$$-\frac{c_i}{b_{i-M}}c_{i-M}x_{i-2M} + \left(b_i - \frac{c_i}{b_{i-M}}a_{i-M}\right)x_i + a_i x_{i+M} = y_i - \frac{c_i}{b_{i-M}}y_{i-M}. \tag{5}$$

We note that the term with respect to x_{i-M} does not appear in (5). Similarly, it follows from (3) and (4) that

$$c_i x_{i-M} + \left(b_i - \frac{a_i}{b_{i+M}}c_{i+M}\right)x_i - \frac{a_i}{b_{i+M}}a_{i+M}x_{i+2M} = y_i - \frac{a_i}{b_{i+M}}y_{i+M}, \tag{6}$$