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Fundamental kernel-based method for backward space–time fractional diffusion problem

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ABSTRACT

Based on kernel-based approximation technique, we devise in this paper an efficient and accurate numerical scheme for solving a backward space–time fractional diffusion problem (BSTFDP). The kernels used in the approximation are the fundamental solutions of the space–time fractional diffusion equation expressed in terms of inverse Fourier transform of Mittag-Leffler functions. The use of Inverse fast Fourier transform (IFFT) technique enables an accurate and efficient evaluation of the fundamental solutions and gives a robust numerical algorithm for the solution of the BSTFDP. Since the BSTFDP is intrinsic ill-posed, we apply the standard Tikhonov regularization technique to obtain a stable solution to the highly ill-conditioned resultant system of linear equations. For choosing optimal regularization parameter, we combine the regularization technique with the generalized cross validation (GCV) method for an optimal placement of the source points in the use of fundamental solutions. Meanwhile, the proposed algorithm also speeds up the previous method given in Dou and Hon (2014). Several numerical examples are constructed to verify the accuracy and efficiency of the proposed method.

1. Introduction

In the last decade, partial differential equations of fractional orders have become the focus of many research studies due to its potential applications in modelling real physical phenomena from numerous diverse and widespread fields in fluid mechanics, visco-elasticity, biology, physics, engineering and finance. Fractional calculus in mathematical point of view is a natural extension of integer-order calculus. It had successfully modelled some physical processes arisen from real-life problems, for instance, the modelling on the transport of passive tracers carried by fluid flows in a porous medium under groundwater hydrology. Studies of the complicated phenomena of the interstitial fluid flows in relation to fractional orders are still under intensive researches and particularly challenging for quantitative analyses and modelling. A space–time fractional diffusion equation, $\frac{\partial^\beta}{\partial t^\beta}u(\mathbf{x},t)+(-\Delta)^{\frac{\alpha}{2}}u(\mathbf{x},t)=0$, obtained from the standard diffusion equation $\frac{\partial}{\partial t}u(\mathbf{x},t)-\Delta u(\mathbf{x},t)=0$ by replacing the second order space-derivative by a fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$, $1<\alpha\leq 2$ and the first order time-derivative by a fractional derivative of order $\beta>0$ (in Caputo or Riemann–Liouville sense), has higher adaptability in modelling from the view point of physical applications. In general, fractional derivative in time can be used to describe particle sticking and trapping phenomena whereas fractional space derivative is more appropriate to simulate

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long particle jumps. These complicated physical processes produce a concentration profile with sharper peaks and heavier tails. In particular when $\beta = 1$, the space-fractional diffusion equation (SFDE) models particle motion in a heterogeneous environment in which the probability of long particle jumps follows a power law [1,2].

Due to the demand on a good estimation of heat temperature and heat flux history from only spatially observed data during a heat propagation process, the investigation of backward heat conduction problem (BHCP) is also important in many branches of engineering sciences. Although heat conduction process is very smooth, the process is irreducible. This means that the characteristic of the solution (for instance, the shape of the interior heat flow) may not be affected by the observed data. On the other hand, the heat conduction process has no finite propagation speed and thus an efficient non-destructive testing technique can be achieved at a comparably much lower cost. For instance, in modelling transient heat conduction phenomena by the standard parabolic heat conduction equation, a complete recovery of the unknown solution is attainable from solving a well-posed problem providing the initial temperature distribution and boundary conditions are given. In real-life application, however, the boundary conditions are usually missing and temperature distribution data can only be measured with noise in some scattered spatial points at a particular time. This makes the BHCP in nature unstable because the unknown solution and its derivatives have to be determined from indirect observable data which contain measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix. The BHCP is a typical ill-posed problem in the sense that the solution of BHCP does not continuously depend on the final temperature data. Any small change in the given final temperature data may induce enormous change in the solution. The lack of mathematical analysis and efficient algorithm hinders the development of developing low cost and efficient non-destructive testing technique in real applications.

Based on recently developed kernel-based approximation method, we aim at developing efficient and accurate numerical algorithm for solving this typical kind of ill-posed BSTFDP. Numerical methods for solving well-posed initial/boundary value problems of fractional diffusion equation can be found from the recent works of Wen and Hon [3], Brunner et al. [4], Cuesta and Palencia [5], Li and Xu [6], Liu et al. [7], Meerschaert et al. [8] and Yang et al. [9]. Due to the ill-posedness of the BSTFDP, some kinds of regularization techniques are essential to obtain a stable reconstruction of the solution. For BHCPs, stable approximation by using regularization techniques can be found in the works of Han et al. [10] and Muniz et al. [11]. Recently, numerical solutions were given respectively by Hon and Li [12], Liu [13], Mera [14] and Wei and Wang [15] by using the Method of Fundamental Solutions (MFS). It is well known that the accuracy of the MFS depends on a suitable placement of source points. Mera in [14] placed the source points on a line below the initial time whereas Hon and Li in [12] obtained an improved solution by placing the source points uniformly over both the temporal and spatial axes. Wei and Wang in [15] provided a new choice method for the source points by using single layer heat potential.

The backward problem of TFDE was tackled respectively by Liu [16] using quasi-reversibility method and Ren et al. [17] by spectral truncation method. For inverse problems of space–time fractional diffusion equation in one-dimension, some numerical solutions have been given by Aldoghaither et al. [18]; Wei et al. [19] and Zheng and Wei [20]. To the knowledge of the authors, there are still very few numerical algorithms for solving inverse problems or backward problems of SFDE and STFDE in higher-dimensional domain. Based on our previous work [21] on solving the backward problem of TFDE for the special case of time fractional derivative $\beta = 2/3$ by using the method of fundamental solutions, we establish the numerical construction of solution for BSTFDP for general cases of order of the temporal fractional derivative by using inverse Fourier transform of the Mittag-Leffler functions. For efficient computation, we apply the IFFT technique to evaluate the fundamental solutions. Since it is impossible to use IFFT directly to obtain the numerical value of fundamental solution at each point due to the definition of FFT and CPU time complexity, we devise in this paper a feasible and flexible strategy to obtain a stable, efficient, and accurate solution to the BSTFDP. To solve the highly ill-conditioned resultant system of linear equations in our computation, we adapt the use of the standard Tikhonov regularization technique. Motivated by our recent works in [22,21] expressing the solution in terms of integrals of Green's function of Cauchy's problem, we combine the regularization technique with the Generalized Cross Validation (GCV) method for the placement of the source points in the use of fundamental solutions as kernels to choose the optimal regularization parameter. Numerical examples are constructed to verify both the efficiency and accuracy of our proposed method.

This paper is organized as follows. In Section 2 we consider the backward problems of STFDE with the temporal fractional derivative defined in the sense of Caputo and the spatial derivative defined by fractional Laplacian. The numerical scheme based on the kernel-based approximation is devised in Section 3. Since the fundamental solutions are given in terms of the inverse Fourier transform of the Mittag-Leffler function, we use IFFT algorithm in our computation to obtain the numerical values of the fundamental solutions. Numerical verification on the efficiency and accuracy of the proposed method for backward problems of TFDE, SFDE and STFDE is presented in Section 4. Conclusion is given in Section 5.

2. Backward space–time fractional diffusion problem

Consider the following space–time fractional diffusion equation:

$$\frac{\partial^\beta u(\mathbf{x},t)}{\partial t^\beta} + (-\Delta)^{\frac{\alpha}{2}} u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \mathbb{R}^n,\ t \in (0,T), \tag{2.1}$$