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Iteration methods for Fredholm integral equations of the second kind

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Abstract

In this paper, we propose an efficient iteration algorithm for Fredholm integral equations of the second kind. We show that for every step of iteration the coefficient matrix of the linear system to be inverted remains the same as in the original approximation methods, while we obtain the superconvergence rates for every step of iteration. We apply our iteration methods to various approximation methods such as degenerate kernel methods, Galerkin, collocation and new projection methods. We illustrate our results by numerical experiments.

Keywords: Integral equations; Superconvergence rates; Galerkin methods; Collocation methods; Iteration methods

1. Introduction

In this paper we consider the numerical solution of an integral equation of the second kind

$$u - \mathcal{K}u = f, \tag{1.1}$$

where $\mathcal{K}$ is a compact linear operator on a Banach space $\mathbb{X}$ with norm $\|\cdot\|$, and $f \in \mathbb{X}$ is a given function and u an unknown function belonging to $\mathbb{X}$. The corresponding homogeneous equation is assumed to have no non-trivial solution in $\mathbb{X}$.

The Galerkin and collocation methods are the well-established numerical methods for the approximate solution of (1.1) (see [1,7]). It has been shown by Sloan in [15] that the iterated Galerkin and iterated collocation methods provide, in general, more accurate approximations to the solution u than the Galerkin and collocation approximations. This iterated technique was also extended to Petrov–Galerkin methods, discrete Petrov–Galerkin methods, degenerate kernel methods and new projection methods (see [3,4,8,10,12,13]). Iterated Galerkin and iterated collocation methods for the Hammerstein equations were extended by Kaneko and Xu in [11].

Unfortunately, the iteration method described by Sloan [10,15] does not lead to further improvement if it is iterated the second time. The computation of a sufficiently accurate solution may require the use of much finer partition of

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the domain and may thus involve the solution of a correspondingly large linear system. On the other hand, it is well known that the coefficient matrix of an integral equation is dense, and it requires heavy work to generate the coefficient matrix if the partition is much finer (see [5,6]). A reiterated Galerkin method was introduced by Porter and Stirling and improved error bounds obtained for every Sloan iteration (see [14]). However, their method is not suitable to interpolatory projection methods, such as collocation methods, which are more popular in engineering computation.

In this paper we develop iteration algorithms for the Eq. (1.1) with less additional computational efforts for each step of iteration. We show that for every step of iteration the coefficient matrix of the linear system to be inverted remains the same as that in the original approximation method while we can obtain the superconvergence rates for every step of iteration. We show that the additional orders of convergence for every step of iteration are of the order (h^{4r}) in the new projection method and (h^{2r}) in Galerkin, collocation and degenerate kernel methods, where h is the norm of the partition and r is the order of the piecewise polynomials used in the approximation.

We organize the paper as follows: In Section 2, we develop iteration algorithms for the Eq. (1.1) and we give convergence analysis for our iteration algorithms. We apply these iteration methods in Section 3 to four different approximation methods, which are new projection methods, Galerkin methods, collocation methods and degenerate kernel methods, and exhibit our proposed iteration methods having higher rates of convergence. In Section 4, we present numerical examples which illustrate our methods and confirm the theoretical results obtained in Sections 2 and 3.

2. Iteration methods

In this section we develop iteration algorithms for numerical solutions of integral equations of the second kind (1.1) and provide the convergence analysis for our proposed iteration algorithms.

Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. Assume $\mathcal{K}$ is a compact linear operator on $\mathbb{X}$, and $\mathcal{I}$ the identity operator on $\mathbb{X}$. We are interested in the numerical solution of the Fredholm integral equation of the second kind

$$(\mathcal{I}-\mathcal{K})u = f. \tag{2.1}$$

We assume that for any $f \in \mathbb{X}$, Eq. (2.1) is uniquely solvable in $\mathbb{X}$. We let $\{\mathbb{X}_n : n \in \mathbb{N}\}$ be a sequence of finite-dimensional subspaces of $\mathbb{X}$. Assume $\mathcal{K}_n : \mathbb{X} \to \mathbb{X}$ be bounded linear operators which are norm convergent $\|\mathcal{K}_n - \mathcal{K}\| \to 0$ as $n \to \infty$, and $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be a sequence of bounded linear projection operators satisfying $\|\mathcal{P}_n u - u\| \to 0$ as $n \to \infty$ for all $u \in \mathbb{X}$. We can classify the approximation method for (2.1) into two approximation schemes, that is, seeking $u_n, \widetilde{u}_n \in \mathbb{X}$ such that

$$\text{(Scheme I)} \quad u_n - \mathcal{K}_n u_n = f, \tag{2.2}$$

$$\widetilde{u}_n = \mathcal{K}u_n + f, \tag{2.3}$$

and seeking $u_n \in \mathbb{X}_n$ and $\widetilde{u}_n \in \mathbb{X}$ such that

$$\text{(Scheme II)} \quad u_n - \mathcal{P}_n\mathcal{K}u_n = \mathcal{P}_n f. \tag{2.4}$$

$$\widetilde{u}_n = \mathcal{K}u_n + f, \tag{2.5}$$

where iterated approximation solution $\widetilde{u}_n$ is usually called Sloan iterated solution (see [15]). By assumptions, we conclude that both approximation schemes are uniquely solvable, $(\mathcal{I}-\mathcal{P}_n\mathcal{K})^{-1}$ and $(\mathcal{I}-\mathcal{K}_n)^{-1}$ exist and are uniformly bounded for sufficiently large n.

We remark that iterated degenerate kernel methods, new projection methods belong to Scheme I, while Galerkin, Petrov–Galerkin and collocation methods belong to Scheme II. When $u_n \in \mathbb{X}_n$, $\mathcal{K}_n = \mathcal{P}_n\mathcal{K}$ and $f \in \mathbb{X}_n$ in Scheme I, Scheme II coincides with Scheme I.

It is clear that under suitable conditions the iteration scheme (2.3) and (2.5) exhibit global superconvergence, that is, it can converge more rapidly than the rate achieved by the approximation scheme (2.2) and (2.4) (see [10,15]). In order to get much higher rate of convergence, we propose an efficient iteration algorithm having much higher order of convergence, while we need less additional computational efforts for the implementation.

Iteration Algorithm: Set $u_n^{(0)} := \widetilde{u}_n$, for $k = 0, 1, \ldots,$

Step 1 $\widetilde{u}_n^{(k)} := \mathcal{K}u_n^{(k)} + f$;