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Existence and approximation of solution to neutral fractional differential equation with nonlocal conditions

Alka Chadha, D.N. Pandey *

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India

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ABSTRACT

This paper is concerned with the approximation of the solution for neutral fractional differential equation with nonlocal conditions in an arbitrary separable Hilbert space H. We study an associated integral equation and then, consider a sequence of approximate integral equations obtained by the projection of considered associated nonlocal neutral fractional integral equation onto finite dimensional space. The sufficient condition for the existence and uniqueness of solutions to every approximate integral equation is derived by using analytic semigroup and Banach fixed point theorem. We demonstrate convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. Moreover, we consider the Faedo–Galerkin approximations of the solution and demonstrate some convergence results. An example is also provided to illustrate the discussed abstract theory.

1. Introduction

In the recent years, the investigation of fractional differential equation has been picking up much attention of researchers. This is due to the fact that fractional differential equations have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in super sonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors and theory of population dynamics. The fractional differential equation is an important tool to describe the memory and hereditary properties of various materials and phenomena. The details on the theory and its applications may be found in books [1–4] and references therein.

On the other hand, the nonlocal problem for abstract evolution equations has been studied by many authors. The existence of a solution for abstract Cauchy differential equation with nonlocal conditions in a Banach space has been considered first by Byszewski [5]. In physical science, the nonlocal condition may be connected with better effect in applications than the classical initial condition since nonlocal conditions are normally more exact for physical estimations than the classical initial condition. For the study of nonlocal problems, we refer to [6–11] and references given therein.

The Faedo–Galerkin approach may be used for the study of more regular solutions, imposing higher regularity on the data. Also, the Faedo–Galerkin method may be used within a variational formulation in order to provide solutions of the equations under possibly weaker assumptions on the data, and may also prove a very useful tool for numerical approximation of

* Corresponding author.
E-mail addresses: alkachadda23@gmail.com (A. Chadha), dwij.iitk@gmail.com (D.N. Pandey).

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the equations. In [12], the Faedo–Galerkin approximations of the solutions for functional Cauchy problem in a separable Hilbert space have been studied by Miletta with the help of analytic semigroup theory and Banach fixed point theorem. In [13], authors have extended the results of [12] and considered the Faedo–Galerkin approximations of the solutions to a class of functional integro-differential equation. The Faedo–Galerkin approximations of the solutions for nonlinear Sobolev type evolution equation have been studied by authors of [14]. In [8], authors have considered the Faedo–Galerkin approximations of the solutions to neutral functional differential equations with nonlocal conditions. The existence and approximations of solutions for fractional differential equation have been investigated by the author in [15]. The existence and Faedo–Galerkin approximations of the solutions for fractional integral equations has been studied by authors in [16]. In [17], authors have discussed the Faedo–Galerkin approximations of solutions to fractional differential equations with a deviating argument with the help of analytic semigroup. The present work extends the previous studies by examining the neutral fractional differential equation with nonlocal conditions and determining the mild solution for the nonlocal neutral fractional differential equation. For a nice introduction to the existence of an approximate solution and associated study of different problems, we refer to the Refs. [7,18–24,31] and references given therein.

Motivated by of above mentioned work, the main objective of this work is to investigate the Faedo–Galerkin approximations of the solution to the following nonlocal neutral fractional differential equation in a separable Hilbert space $(H, \|\cdot\|, (\cdot,\cdot))$

$$^{c}\mathbf{D}_t^q[y(t) + \mathcal{G}(t, y(t), y(t-h_1))] = -By(t) + \mathcal{F}(t, y(t), y(t-h_2)), \quad t \in [0, T], \tag{1.1}$$

$$g(y) = \psi, \quad \text{on } [-h, 0],\ h > 0, \tag{1.2}$$

where $0 < q < 1$, $0 < T < \infty$, $^{c}\mathbf{D}_t^q$ is the fractional derivative in Caputo sense and $h = \max\{h_1, h_2\}$, $h_1, h_2 > 0$. In (1.1), $B : H \supset D(B) \to H$ is assumed to be a closed, self adjoint and positive definite linear operator with dense domain $D(B)$ such that $-B$ is the infinitesimal generator of an analytic semigroup of bounded linear operator on H. The functions $\mathcal{G}, \ \mathcal{F} : [0, T] \times H \times H \to H$, $g : C([-h, 0]; H) \to C([-h, 0]; H)$ are nonlinear continuous functions satisfying certain conditions to be mentioned later. Neutral differential equation arises in many areas of applied mathematics, science and engineering such as the theory of aeroelasticity [25] and lossless transmission lines [26]. The theory of heat conduction in materials and the lumped control systems can be described by neutral differential equations. The system of rigid heat conduction with finite wave spaces can be modeled in the form of the integro-differential equation of neutral type with delay. For the initial study of the neutral functional differential equations with finite delay, we refer to books [27,28] and references given therein.

The article is organized as follows: Section 2 provides some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. Section 3 derives the sufficient condition for the existence and uniqueness of the approximate solutions by using analytic semigroup and Banach fixed point theorem. Section 4 proves the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and Section 5 focuses on the convergence of the approximate Faedo–Galerkin solutions. Section 6 presents an example.

2. Preliminaries and assumptions

In this section, some basic definitions, preliminaries, theorems and lemmas and assumptions which will be used to prove existence result, are provided.

Throughout the work, we assume that $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is a separable Hilbert space. The symbol $C([0, T]; H)$ stands for the Banach space of all the continuous functions from $[0, T]$ into H equipped with the norm $\|z\|_C = \sup_{t\in[0,T]} \| z(t)\|$ and $L^p((0, T); H)$ denotes the Banach space of all Bochner-measurable functions from $(0, T)$ to H with the norm

$$\|z\|_{L^p} = \left(\int_{(0,T)} \|z(s)\|^p ds\right)^{1/p}, \quad z \in L^p((0, T); H).$$

In this work, $-B$ is assumed to be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{\mathcal{T}(t) : \ t \geq 0\}$ on H. Therefore, there exist constants $C \geq 1$ and $\delta \geq 0$ such that $Ce^{\delta t} \geq \|\mathcal{T}(t)\|$ for $t \geq 0$. In addition, we note that

$$\left\|\frac{d^j}{dt^j}\mathcal{T}(t)\right\| \leq M_j, \quad t > t_0,\ t_0 > 0,\ j = 1, 2, \ldots, \tag{2.1}$$

where M_j are some positive constants. Henceforth, without loss of generality, we may assume that $\mathcal{T}(t)$ is uniformly bounded by M i.e., $\| \mathcal{T}(t)\| \leq M$ and $0 \in \rho(-B)$ i.e., $-B$ is invertible. This permits us to define the positive fractional power B^η as closed linear operator with domain $D(B^\eta) \subseteq H$ for $\eta \in (0, 1]$. Moreover, $D(B^\eta)$ is dense in H with the norm

$$\|y\|_\eta = \|B^\eta y\|, \quad \forall\, y \in D(B^\eta). \tag{2.2}$$

Hence, we signify the space $D(B^\eta)$ by H_η endowed with the η-norm $(\|\cdot\|_\eta)$. Also, we have that $H_\kappa \hookrightarrow H_\eta$ for $0 < \eta < \kappa$ and therefore, the embedding is continuous. Then, we define $H_{-\eta} = (H_\eta)^*$, for each $\eta > 0$ and the dual space of H_η, is a Banach space with the norm $\|z\|_{-\eta} = \|B^{-\eta}z\|$ for $z \in H_{-\eta}$. For more study on the fractional powers of closed linear operators, we refer to book by Pazy [29].