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A wavefront-based Gaussian beam method for computing high frequency wave propagation problems

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ABSTRACT

We present a novel wavefront method based on Gaussian beams for computing high frequency wave propagation problems. Unlike standard geometrical optics, Gaussian beams compute the correct solution of the wave field also at caustics. The method tracks a front of two canonical beams with two particular initial values for width and curvature. In a fast post-processing step, from the canonical solutions we recreate any other Gaussian beam with arbitrary initial data on the initial front. This provides a simple mechanism to include a variety of optimization processes, including error minimization and beam width minimization, for a posteriori selection of optimal beam initial parameters. The performance of the method is illustrated with two numerical examples.

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1. Introduction

In direct discretization methods for high frequency wave problems, a large number of grid points is needed to resolve the wave oscillations, and the computational cost to maintain constant accuracy grows algebraically with the frequency. At sufficiently high frequencies, direct simulations are not feasible. As an alternative, one can use high frequency asymptotic methods where the cost is either independent of or grows slowly with the frequency, see [1,2]. The Gaussian beam method is one such asymptotic method for computing high frequency wave fields in smoothly varying inhomogeneous media. It was proposed by Popov [3], based on earlier work of Babič and Pankratova [4]. The method was first applied by Katchalov and Popov [5], Červený et al. [6] and Klimeš [7] to describe high-frequency seismic wave fields by the summation of Gaussian beams. In quantum chemistry, Gaussian beams are higher order versions of classical coherent states, and they are used to approximate the Schrödinger equation; see e.g. Heller, Herman and Kluk [8,9]. Gaussian beams were later applied to seismic migration by Hill [10,11]. For a rigorous mathematical analysis of Gaussian beams we refer to [12] and the more recent investigations on accuracy [13–18]. The main advantage of this method is that Gaussian beams provide the correct solution also at caustics where standard geometrical optics breaks down.

In the Gaussian beam method, the initial/boundary data or the wave sources which generate the high frequency wave field are decomposed into Gaussian beams. Individual Gaussian beams are computed in a Lagrangian fashion by ray tracing, where quantities such as the curvature and width of beams are calculated from ordinary differential equations (ODEs) along the central ray of the beams. The initial conditions for the ODEs are obtained from the field decomposition at the boundary or

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the source. The contributions of the beams concentrated close to their central rays are determined by Taylor expansion. The wave field at a receiver is then obtained as a weighted superposition of the Gaussian beams situated close to the receiver.

The past few years have seen a renewed interest in Gaussian beam based methods and their applications [19–22]. One new direction is the Eulerian Gaussian beam summation methods [23–26]. In this approach, the problem is formulated by Liouville-type equations in phase space giving uniformly distributed Eulerian traveltimes and amplitudes for multiple sources. A recent survey of Gaussian beam methods can be found in [27]. Numerical approaches for treating general high frequency initial data for superposition over physical space were considered in [28,29] for the wave equation.

In this paper, we revisit the Lagrangian formulation and present a wavefront method for computing Gaussian beams. Wavefront methods have been very successful for standard geometrical optics as they provide a simple mechanism for controlling the resolution and accuracy of the numerical approximation [30,31]. Using them with Gaussian beams is not as straightforward since the beam method strongly depends on the distribution and width of the beams at the initial front and on how they spread during their evolution, see e.g. [17,32,33]. We construct our novel wavefront method based on two canonical functions. We present an efficient strategy consisting of two parts: (1) We compute the wavefronts together with a set of canonical solutions with a priori and fixed initial data; and (2) In a post-processing step, from the canonical solutions we recreate Gaussian beams with a posteriori, optimal selection of initial data and compute the wave field by a weighted sum of beams. This strategy has a few advantages. First, we can compute beams with any arbitrary initial conditions by a simple linear combination of the canonical solutions at no extra cost. Second, our wavefront construction provides a simple mechanism to include a variety of optimization processes, e.g. error minimization, for a posteriori selection of optimal initial parameters. Finally, since the geometrical optics solution can be recovered by the first set of canonical solutions, it is possible to design an efficient hybrid method which switches between the geometrical optics (which does not require the post-processing step) and Gaussian beam solutions smoothly. We present numerical examples to verify the efficiency, accuracy, and the flexibility of the algorithm.

The first step of our algorithm in part 1, which is the computation of wavefronts, is an adaptation of the front tracking scheme in [34]. It is to be noted that in order to control the resolution of wavefronts, we can also adapt and include other front tracking methods, such as the grid-based particle method [35] and the fast interface tracking method [36,37], in the algorithm. The main contributions of this paper include the second step of the algorithm in part 1, i.e. the construction of canonical functions, and the fast post-processing technique in part 2 based on an optimal selection of the beams' initial data.

The rest of the paper is organized as follows. In Section 2 we first review the Gaussian beam models for the computation of time harmonic high frequency waves (Sections 2.1–2.4). We then present and discuss different choices of initial parameters in the computation of Gaussian beams (Section 2.5). Next, in Section 3 we describe the new wavefront method based on Gaussian beam summation and canonical functions. Numerical examples are performed in Section 4. Finally, we summarize our conclusions in Section 5.

2. Gaussian beam models

Gaussian beams are asymptotic solutions of linear wave equations. They can also be extended to some dispersive wave equations like the Schrödinger equation. Gaussian beam summation is an approximate model for linear high frequency wave propagation problems. In this approach, the initial/boundary data are decomposed into individual Gaussian beams, which are computed by a system of ODEs along their central rays. The contribution of each beam close to its central ray is approximated by Taylor expansion. The wave field is then obtained by summing over the beams. In this section, we review the governing equations for computing Gaussian beams and formulate the beam summation model.

2.1. High frequency waves and asymptotic approximations

We start with the scalar wave equation

$$v_{tt}(t, \boldsymbol{x}) - c(\boldsymbol{x})^2 \, \Delta v(t, \boldsymbol{x}) = 0, \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{1}$$

where $v = v(t, \boldsymbol{x})$ is the wave solution, t and $\boldsymbol{x} = (x, y)^\top$ are the temporal and spatial variables, respectively, and $c(\boldsymbol{x})$ is the local speed of wave propagation in the medium. We complement the wave equation (1) with highly oscillatory initial data that generate high-frequency solutions. The exact form of the data will not be important here, but a typical example is $v(0, \boldsymbol{x}) = a(\boldsymbol{x}) \, \exp(i \omega \, \boldsymbol{k} \cdot \boldsymbol{x})$, where $\omega \gg 1$ is the angular frequency and $|\boldsymbol{k}| = 1$. We assume that the wavelength, which is inversely proportional to ω, is much smaller than the typical scale of the medium structure (variations in the wave speed) and the wave propagation distance (the size of the computational domain). Hence, we encounter a multiscale problem with highly oscillatory solutions. Note that with slight modifications, the techniques we describe here will also carry over to systems of wave equations, such as the Maxwell and elastodynamic equations.

We consider time-harmonic waves of type

$$v(t, \boldsymbol{x}) = u(\boldsymbol{x}) \, \exp(i \omega \, t). \tag{2}$$