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Two symbolic algorithms for solving general periodic pentadiagonal linear systems☆

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ABSTRACT

In this paper, we present two novel symbolic computational algorithms to solve periodic pentadiagonal (PP) linear systems. These two algorithms are based on a special matrix decomposition and the use of any fast pentadiagonal linear solver, respectively. Some numerical examples are given in order to demonstrate the performance of the proposed algorithms and their competitiveness with existing algorithms. All of the experiments are performed on a computer workstation with the aid of programs written in MATLAB.

1. Introduction and objectives

Consider a general periodic pentadiagonal (PP) linear system of the form

$$A\mathbf{x} = \mathbf{f}, \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, $\mathbf{f} = (f_1, f_2, \dots, f_n)^T$ are unknown and known vectors of length n, respectively, and the coefficient matrix A is an n-by-n periodic pentadiagonal matrix takes the following form

$$A = \begin{pmatrix} d_1 & a_1 & \tilde{a}_1 & & b_1 \\ b_2 & d_2 & a_2 & \ddots & \\ \tilde{b}_3 & b_3 & \ddots & \ddots & \tilde{a}_{n-2} \\ & \ddots & \ddots & \ddots & a_{n-1} \\ a_n & & \tilde{b}_n & b_n & d_n \end{pmatrix}. \tag{1.2}$$

Here, the superscript symbol T corresponds to the transpose operation of vector or matrix. Throughout this paper, we assume that the coefficient matrix A is nonsingular. This special type of system appears not only in a variety of theoretical areas (in linear algebra or numerical analysis), but also in different areas of applications. For example, such system frequently arises in spline approximation, data interpolation, the numerical solution of ordinary and partial differential equations (ODEs and PDEs), vision, image and signal processing (VISP), fluid mechanics, and parallel computing, etc. [1–11].

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Based on the partition procedure described in [9], system (1.1) can be also written in the following two-by-two block form

$$\begin{pmatrix} P & \mathbf{u} \\ \mathbf{v}^T & d_n \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ x_n \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}} \\ f_n \end{pmatrix}, \quad (1.3)$$

where $\mathbf{u} = (b_1, \underbrace{0, \dots, 0}_{(n-4)-terms}, \tilde{a}_{n-2}, a_{n-1})^T$, $\mathbf{v} = (a_n, \underbrace{0, \dots, 0}_{(n-4)-terms}, \tilde{b}_n, b_n)^T$, $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_{n-1})^T$, $\tilde{\mathbf{f}} = (f_1, f_2, \dots, f_{n-1})^T$, and P is the $(n-1)$-th leading principal submatrix of A.

A standard solution of the PP linear system (1.1) can be obtained if we know the inverse matrix of A, which would reduce the problem of determining the solution for this system to one of matrix multiplication having the explicit form $\mathbf{x} = A^{-1}\mathbf{b}$. However, the inverse of an n-by-n periodic pentadiagonal matrix usually cannot be obtained in $O(n)$ operations, since the inverse matrix has n^2 elements and it is not, in general, a periodic pentadiagonal matrix itself [12–15]. Therefore, many direct numerical methods for solving the PP linear systems have been investigated in recent years. For example, by using the matrix reformulation and Sherman–Morrison–Woodbury formula, Sogabe gave an efficient computational algorithm for solving PP linear systems in [16]. In paper [17], Karawia presented a recursive algorithm (**KPENTA** algorithm) for the same purpose. The algorithm is based on the LU decomposition that represents the coefficient matrix A as a product of lower and upper triangular matrices $A = LU$. Consequently, solving $A\mathbf{x} = \mathbf{f}$ can be achieved by solutions of two triangular linear systems $L\mathbf{y} = \mathbf{f}$ and $U\mathbf{x} = \mathbf{y}$. However, Karawia's algorithm may suffer from breakdown for some cases.

As an important subset of the PP linear systems, a periodic pentadiagonal Toeplitz (PPT) linear system $\tilde{A}\mathbf{x} = \mathbf{f}$ can be obtained by setting

(1) $a_i = a, \quad b_i = b, \quad d_i = d \quad \text{for } i = 1, 2, \dots, n,$

(2) $\tilde{a}_i = \tilde{a} \quad \text{for } i = 1, 2, \dots, n-1,$

(3) $\tilde{b}_i = \tilde{b} \quad \text{for } i = 2, 3, \dots, n.$

Very recently, in their paper [18], Jia and Jiang have derived a fast numerical algorithm for solving PPT linear systems based on a structure preserving matrix decomposition and Sherman–Morrison–Woodbury formula. All of the algorithms mentioned above solve PP (or PPT) linear systems in linear time. The main contributions of this paper are summarized as follows.

- In order to remove all cases where Karawia's algorithm fails, we propose a symbolic algorithm (Algorithm 2.1) based on a special matrix decomposition (*LDU* decomposition) which is different from the classical *LDU* decomposition given in [19]. The computational cost of Algorithm 2.1 is less than those of two existing algorithms in [17,16].
- The combination of the partition procedure described in [9] and generalized Doolittle *LU* decomposition solver yields another symbolic algorithm (Algorithm 2.2) whose computational cost is also less than those of the algorithms given in [17,16].

Moreover, it should be mentioned that the corresponding results in this paper can be directly applied to periodic anti-pentadiagonal (PAP) linear systems and periodic tridiagonal (PT) linear systems [20].

An outline of this paper is as follows. In the next section, we are going to derive two symbolic computational algorithms to solve system (1.1) in linear time. The main advantage of these two algorithms is that they take less computational costs and will never suffer from break down. In addition, the solution of periodic anti-pentadiagonal (PAP) linear systems is also discussed. In Section 3, several numerical experiments are provided to show the performance and effectiveness of our algorithms. Finally, we make some concluding remarks in Section 4.

2. Main results

In this section, we will formulate two novel symbolic computational algorithms that will not suffer from breakdown to solve the PP linear system as in (1.1).

2.1. A symbolic algorithm for solving PP linear systems

Based on a special matrix decomposition, we describe a method to solve the PP linear systems in this subsection. To do this, we begin by considering the lower triangular matrix L, the upper unitriangular matrix D and the nearly unit matrix U defined by

$$L = \begin{pmatrix} c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ g_2 & c_2 & 0 & \cdots & \cdots & 0 \\ \tilde{b}_3 & g_3 & c_3 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \tilde{b}_{n-1} & g_{n-1} & c_{n-1} & 0 \\ 0 & \cdots & 0 & \tilde{b}_n & g_n & c_n \end{pmatrix}, \quad (2.1)$$